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Groundwater drainage flow in a soil layer resting on an inclined leaky bed

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Abstract. We study subsurface storm flow from a planar hill slope, a problem that is similar hydraulically to lateral flow toward drains in landfills. Our analysis is based on the linearized one-dimensional Boussinesq equation (Dupuit-Forchheimer approximation), which is extended to allow for leakage through the underlying barrier. This linear advection-diffusion equation has a greater range of validity than the kinematic wave equation. Stating it in terms of the discharge, the variable of primary hydrologic interest, we integrate it numerically, using an adaptation of the Muskingum-Cunge routing scheme. A single-step computation of the outflow hydrograph, which combines the convenience of an analytical solution formula with acceptable accuracy, is proposed as a design tool and as a means of parameterization of drainage from hill slopes. Depth profiles are determined afterwards by a simple integration of Darcy's law. Examples of the buildup and recession phases, with and without leakage, demonstrate the application of the computational method.

1. Introduction

The hydraulics of drainage from a soil layer resting on an inclined bed of lesser conductivity is of interest in watershed modeling and in landfill engineering (lateral flow toward leachate collection drains installed above liners). The response to rainfall of a near-ground soil zone, called interflow, can contribute significantly to runoff to streams; in the case of heavy infiltration, *Beven* [1981] has termed this runoff "subsurface storm flow." *Henderson and Wooding* [1964] and *Childs* [1971] derive the extended Boussinesq equation by formulating subsurface storm flow on a wide, planar layer, resting on an inclined impervious base within the Dupuit-Forchheimer theory of unconfined flow. In the Dupuit approximation the pressure distribution is hydrostatic; hence the potential and the velocity are constant over the depth. Here the infiltration rate is assumed to be known, and the difficulty of a rigorous solution of unsaturated-saturated flow is evaded. This simplification, as well as the geometric schematization, are justified as means of parameterization of the solution for inclusion in watershed models [*Brutsaert*, 1994] and for landfill drainage calculations.

No general analytical solution is known for the nonlinear problem. *Van de Giesen et al.* [1994] solve the corresponding two-dimensional (2-D) Laplace equation analytically but for a horizontal bed and after linearization of the free-surface condition. *Sanford et al.* [1993] fit an empirical formula to experimental data from the drainage phase. Numerical solutions for the case of a wide, planar hill slope with infiltration have been published by *Beven* [1981] and by *Pi and Hjelmfeld* [1994]. *Beven* [1981] first computed the depth profiles and then used them to determine the outflow rate from a storage balance over the plane, a procedure also adopted by *Koussis and Lien* [1982] and by *Pi and Hjelmfeld* [1994]. In HELP [*Schroeder et al.*, 1983, and amendments], the lateral flow to drains is estimated by a best-fit formula that attempts to summarize numerical simulation results empirically.

The outflow is indeed the quantity of practical interest in hill slope drainage and in lateral drainage in landfills; however, an indirect discharge solution is prescribed by the nonlinear hydraulics. In contrast, direct computation of the discharge is possible with the linear advection-diffusion (LAD) equation that is derived through linearization of the extended Boussinesq equation. Linearization is predicated on the pressure gradient being small relative to the hill slope; if the pressure gradient is negligible, the flow is of the kinematic wave type [*Beven*, 1981]. The LAD model is useful for analyzing the essential flow features because Boussinesq's equation is mildly nonlinear. An indication of the adequacy of linearization is the close agreement between steady state depth profiles predicted by the nonlinear [*Beven*, 1981] and the linear theories [*Koussis and Lien*, 1982; *Koussis*, 1992] (departures from nonlinearity are strongest at steady state, when depth variation is largest). *Brutsaert* [1994] gives an infinite trigonometric series solution of the linear drainage problem (without infiltration), as does *Chapman* [1995] (linearizing for h^2 and including constant infiltration).

We propose a versatile numerical solution methodology that handles arbitrary inputs simply and efficiently. Taking advan-

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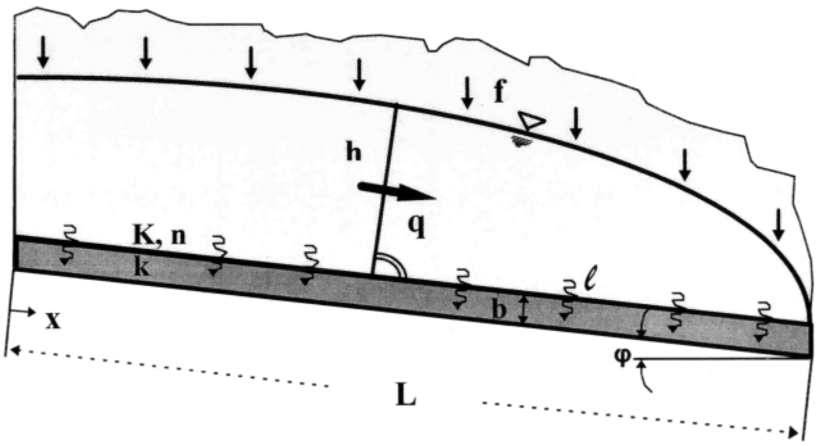

Figure 1. Cross-sectional schematic of a hill slope aquifer, with definitions of symbols.

tage of the ability to formulate LAD equations for either the discharge or the depth, we obtain the discharge directly; the depth solution is calculated from it, if needed. This sequence is analogous to Muskingum flood routing, which models the linear diffusive wave with second-order accuracy, efficiently and robustly [*Cunge*, 1969]. Extending the Muskingum-Cunge scheme to account for leakage, we also derive a one-step, quasi-analytical solution for the outflow at the foot of the hill that can be useful in the design of landfills and in the parameterization of drainage from hill slopes.

2. Governing Equations

The governing equations for subsurface storm flow have been derived by *Henderson and Wooding* [1964] and by *Childs* [1971]. We merely state them here, with reference to Figure 1, amending the mass balance to include leakage, l, through a semipervious base. The discharge per unit width, q, is the product of the specific discharge (Darcy's law) and the depth, h, perpendicular to the bed:

$$q = h(KS - K \cos \phi \partial h/\partial x) \tag{1}$$

where K is the hydraulic conductivity and $S = \sin \phi$, with ϕ the inclination of the base layer against the horizontal, both assumed constant; x is measured along the base, with origin at the top and with $x = L$ at the foot. The equation of *Boussinesq* [1877] is derived from the substitution of (1) into the one-dimensional mass conservation equation:

$$n\ \partial h/\partial t + \partial q/\partial x = f - l \tag{2}$$

$$n\ \partial h/\partial t + KS\ \partial h/\partial x = \cos \phi K\ \partial(h\ \partial h/\partial x)/\partial x + f - l \tag{3a}$$

$$n\ \partial h/\partial t + KS\ \partial h/\partial x = \cos \phi K h\ \partial^2 h/\partial x^2 + \cos \phi K\ (\partial h/\partial x)^2 + f - l \tag{3b}$$

where f is infiltration rate and l is leakage rate, both referenced to a unit area parallel to the base; n is specific yield (effective or kinematic porosity); and t is time.

For a small depth variation the slope of the free surface relative to the base is small, justifying neglecting the term $\cos \phi K\ (\partial h/\partial x)^2$ in (3b) and approximating $\cos \phi\ \partial(h \partial h/\partial x)/\partial x$ by $h_o\ \partial^2 h/\partial x^2$, where h_o is a reference depth in which $\cos \phi$ has been absorbed. Division by n leads to a LAD equation with velocity $u = KS/n$ and hydraulic diffusion coefficient $D = Kh_o/n$,

$$\partial h/\partial t + (KS/n)\ \partial h/\partial x = (Kh_o/n)\ \partial^2 h/\partial x^2 + f/n - l/n \tag{4}$$

A consistent linearization of (1), which we shall use in order to develop the counterpart of (4) in terms of the discharge per unit width, is

$$q = hKS - h_o K\ \partial h/\partial x \tag{5}$$

We now write (4) in the nondimensional form

$$\partial H/\partial T + \partial H/\partial X = H_o\ \partial^2 H/\partial X^2 + F - L_f \tag{6}$$

by introducing the normalized variables [*Koussis*, 1992]

$$X = x/L \qquad H = h/(LS) \qquad T = tKS/(nL)$$
$$F = f/(KS^2) \qquad L_f = l/(KS^2) \tag{7}$$

This nondimensionalization differs from that of *Henderson and Wooding* [1964] (adopted by *Beven* [1981] and by *Koussis and Lien* [1982]) in that the depth is scaled by the height LS, rather than by $LS/2$. The present normalized infiltration rate is thus one quarter of its counterpart of *Henderson and Wooding* [1964]. The nondimensional form of (5) reads, with $Q = q/fL$,

$$QF = H - H_o \partial\ H/\partial X \tag{8}$$

The LAD equation for the discharge q is obtained by differentiating (5) with respect to t, writing $(\partial/\partial t)\partial h/\partial x = (\partial/\partial x)\partial h/\partial t$, and by substituting for $\partial h/\partial t$ from (2), to obtain

$$\partial q/\partial t + (KS/n)\ \partial q/\partial x = (Kh_o/n)\ \partial^2 q/\partial x^2 + [(KS/n) - (Kh_o/n))\ \partial/\partial x](f - l) \tag{9a}$$

When f and l are assumed independent of x and the variables are normalized, the result is

$$\partial Q/\partial T + \partial Q/\partial X = H_o\ \partial^2 Q/\partial X^2 + 1 - f/l \tag{9b}$$

Note that although (9b) is strictly valid for constant f and l, it may serve as an approximation for conditions with variable infiltration. In such cases a function replaces unity on the right-hand side, and q is normalized to a reference infiltration rate.

3. Initial and Boundary Conditions

For a base that is dry at $t = 0$ the initial conditions for the depth and the discharge are

$$h(x, 0) = H(X, 0) = 0 \tag{10a}$$

$$q(x, 0) = Q(X, 0) = 0 \tag{10b}$$

The boundary conditions are quite different for the two formulations. Moreover, specification of physically meaningful conditions at $X = 1$ remains a problem. At $x = X = 0$ the flow is zero,

$$q(0, t) = Q(0, T) = 0 \tag{11a}$$

which translates, via (5), into

$$\partial h/\partial x = h(0, t)/h_o, \text{ or } \partial H/\partial X = H(0, T)/H_o. \tag{11b}$$

At $x = L$ ($X = 1$) the depth is not known, unless the outflow section is submerged, as for example in a case studied by *Henderson and Wooding* [1964]. S. N. Numerov [*Harr*, 1962], applying conformal mapping to the problem of steady flow from a constant-head reservoir into a drain placed at the end of a horizontal, impervious base, showed that the free surface slope at the drain is -1. *Wooding* [1966] found excellent agreement between solutions by conformal mapping and with the Dupuit-Forchheimer approach for steady flow over a 30° slope to a drain (for a specific infiltration rate), except near the boundaries. *Van de Giesen et al.* [1994] showed the seepage face to be a second-order effect. *Koussis* [1992] analyzed linear subsurface storm flow on planar hill slopes using a zero-depth downgradient boundary condition and derived a steady state depth profile with a slope of $-F/H_o \approx -2$ at $X = 1$. *Brutsaert* [1994], in his analytical determination of the drainage (no infiltration) from a planar hill slope, also used zero depth at $X = 1$, stating that this was done for convenience since "in hilly terrain, torrential streams tend to be shallow, and they usually have no effect on the water table position and the flow in adjoining hillslopes" (p. 2759).

Incomplete knowledge of the depth at $X = 1$ indeed does not cause great concern when the flow is driven mainly by gravity, for in that case the "range of influence" of the lower boundary condition is limited. This can be verified analytically. For example, *McEnroe*'s [1993] nonlinear solution for the maximum steady state depth over a sloping landfill liner, which uses Numerov's result to determine the flow depth at $X = 1$, is for all practical purposes numerically the same as the linear solution [*Koussis*, 1992], which uses $H(1) = 0$ and gives $dH/dX \approx -2$ at $X = 1$. The work reported here uses zero depth at $X = 1$:

$$h(L, t) = H(1, T) = 0 \tag{12}$$

Specification of a nonzero $H(1, T)$ would require prescription of the evolution of the unknown depth. The free drainage case, (12), can be defended as an approximation to an unknown yet certainly small depth; but it gives rise to a mathematical oddity. If the outflow is determined from (5), the dominant gravity term is eliminated, leaving $q = -h_o K \partial h/\partial x$. This evaluates properly to $q = fL$ when the analytic steady state solution for H is used, but (5) is not a reliable basis for the determination of the outflow from depth profiles, as it entails the numerical evaluation of $\partial h/\partial x$. A more robust procedure is to obtain the outflow from the storage balance over the slope [*Beven*, 1981; *Koussis and Lien*, 1982; *Pi and Hjelmfeld*, 1994].

Definition of the outflow boundary condition is not straightforward. Consistency with (12) is obtained by setting $\partial h/\partial t = 0$ in the mass conservation equation, (2), whence

$$\partial q/\partial x = f - l, \qquad \text{or} \quad \partial Q/\partial X = 1 - l/f \tag{13}$$

This condition, with $l = 0$ and for a step input of infinite duration, yields the kinematic solution $Q(X) = X$ at steady state, which our linear diffusive wave calculations confirm. In contrast, prescription of the condition that the discharge approaches maximum at $X = 1$, that is, $\partial Q/\partial X = 0$, implies, by (2), that $\partial H/\partial T = 1$ and yields the kinematic wave result $H(1, T) = T$ (valid for $T \leq 1$).

Finally, in order to use the LAD models, the value of h_o must be known. In the absence of leakage, *Koussis* [1992] estimates its value from the steady state depth profile as

$$H_o = h_o/LS = [(1 + 2F^2)^{1/2} - 1]/2F \tag{14}$$

By interpreting F as the effective, or net, normalized infiltration rate, $F_{\text{net}} = F - L_f$, this estimate can be used also in the case of leakage, with further analysis. By applying Darcy's law across a barrier of thickness b and hydraulic conductivity k and by postulating that the average head differential across the barrier is $(h_o + b)$, we obtain the steady and uniform leakage rate

$$l = k(1 + h_o/b) \tag{15}$$

In landfills k is low by design, so $l \ll f$, and therefore it suffices to merely estimate l. If $h_o/b \ll 1$, then $l \approx k$ and $F_{\text{net}} = F - L_f$ is known. If $(h_o/b) \approx 0.1$, say, a direct estimation is adequate; otherwise an iterative solution is required. Consider, for instance, a drain spacing of 100 m, or $L = 50$ m, and a liner thickness $b = 2$ m and slope $S = 0.1$; for $F = 0.2$ and $H_o \approx F/2 = 0.1$ (see below), $h_o = H_o LS \approx 0.5$ m and $h_o/b = 0.25$, whence a single iteration suffices. A cruder estimate for h_o is developed from the kinematic wave approximation, again by interpreting F as the net infiltration rate $F_{\text{net}} = F - L_f$. The nondimensional steady state profiles are straight lines $H(X) = FX$ [*Beven*, 1981], so $H_o = F/2$. This result follows also from (14) directly, upon expanding $(1 + 2F^2)^{1/2}$ in a truncated binomial series (good up to $F \approx 0.5$).

4. The Numerical Solution

Numerical methods can accommodate arbitrary inputs and are thus well suited for evaluating the long-term operation of a landfill or for analyzing the response of a hill slope. For the purposes of testing the numerical solution that we present shall be run with a generic uniform infiltration of constant intensity and finite duration. A variety of methods can be brought to bear on the numerical solution of (9b), provided they can handle the difficulty posed by the dominance of the gravity-driven flow over the flow driven by the depth gradient. That this situation is typical in hill slope hydrology is underscored by the use of the kinematic wave approximation as a computational alternative [*Beven*, 1981]. One such method is based on the very efficient Muskingum scheme of flood routing, the applicability of which to the subsurface storm flow problem follows from the similarity of (9a) to the linear diffusive flood wave equation [e.g., *Cunge*, 1969]. The similarity is highlighted by considering an infinitely wide channel, with discharge per unit width, q, as in (9a): the wave celerity c corresponds to the linear pore velocity due to gravity KS/n, and the hydraulic diffusivity to h_oKS/n.

The Muskingum-Cunge flood routing method is well known [*Cunge*, 1969], requiring only a brief exposition; it has been also applied to mass transport problems [*Koussis*, 1983; *Koussis et al.*, 1983; *Koussis et al.*, 1990; *Syriopoulou and Koussis*, 1991]. The Muskingum-Cunge scheme is derived from the kinematic wave equation by approximating the spatial derivative by a centered-in-time, first-order-in-space-accurate finite difference ratio over ΔX, and the time derivative by a weighted average (spatial weighting coefficient θ) finite difference ratio over ΔT. Extended to include flow exchange, the scheme reads

$$Q_{i+1}(T_{n+1}) = C_1Q_i(T_n) + C_2Q_i(T_{n+1}) + C_3Q_{i+1}(T_n) + (C_1 + C_2)(1 - l/f)\Delta X \tag{16a}$$

where the subscript i indicates the spatial discretization, $X_i = i\Delta X$ and $X_{i+1} = (i + 1)\Delta X$, and the subscript n the temporal discretization, $T_n = n\Delta T$ and $T_{n+1} = (n + 1)\Delta T$. The coefficients C_t are functions of θ and of $\mathbf{C} = \Delta T/\Delta X$, the Courant number, and are given by

$$C_1 = (\mathbf{C} + 2\theta)/(2 + \mathbf{C} - 2\theta) \tag{16b}$$

$$C_2 = (\mathbf{C} - 2\theta)/(2 + \mathbf{C} - 2\theta) \tag{16c}$$

$$C_3 = (2 - \mathbf{C} - 2\theta)/(2 + \mathbf{C} - 2\theta) \tag{16d}$$

The resulting scheme approximates to order $(\Delta x)^2$ a LAD equation with diffusion coefficient $D_n = c\Delta x(0.5 - \theta)$, or, for the LAD equation of subsurface storm flow,

$$D_n = (KS/n)\Delta x(0.5 - \theta) \tag{17}$$

The numerical scheme solves the physical LAD equation when the numerical and physical diffusion coefficients are matched, via the weighting coefficient θ. For (9b), the match yields

$$\theta = 0.5 - H_o/\Delta X \tag{18a}$$

or, in terms of the grid Peclet number $\mathbf{P} = \Delta X/H_o$,

$$\theta = 0.5 - \mathbf{P}^{-1} \tag{18b}$$

The numerical scheme is explicit and stable for $\theta \leq 0.5$ [*Cunge*, 1969] and thus unconditionally stable for physically realizable systems ($D > 0$). Constraints are placed on the grid parameters ΔX and ΔT to ensure that the results are physically plausible. In the absence of infiltration and leakage, these constraints require $C_t \geq 0$, in order to eliminate unphysical oscillations: (1) in response to a positive input, $C_2 \geq 0$, and (2) upon cessation of input, $C_3 \geq 0$ [*Bowen et al.*, 1989]. These grid design constraints (θ, $\mathbf{C}$) or ($\mathbf{P}$, $\mathbf{C}$) are summarized as follows

$$2\theta \leq \mathbf{C} \leq 2 - 2\theta \tag{19a}$$

$$1 - 2\mathbf{P}^{-1} \leq \mathbf{C} \leq 1 + 2\mathbf{P}^{-1} \tag{19b}$$

In the presence of leakage and infiltration, (19a) and (19b) are relaxed; we have used them as conservative guides in grid design. In addition, to ensure that the premise on which the scheme's validity rests is satisfied, that is, gravity-dominated flow, $\mathbf{P} \geq 2$ should be used, although the scheme is robust enough to function with $\mathbf{P}$ slightly less than 2. Given the range of typical F values, say $0.125 \leq F \leq 0.5$, and using $H_o \approx F/2$, it follows that the limit $\mathbf{P} = 2$ allows only a coarse spatial discretization. The discharge profiles are smooth, however; therefore the grid restriction is not severe. In fact, as the applications will demonstrate, computation with a single space step, $\Delta X = 1$, yields useable outflow hydrographs, thus lending the convenience of an analytical solution. The computation proceeds downslope and, since the scheme derives from a first-order equation, does not require imposition of a boundary condition at $X = 1$.

Note further that the kinematic wave solution can be also obtained from (16) by setting $\theta = 0.5$ ($D_n = 0$) and $\mathbf{C} = 1$. This choice of parameters gives

$$Q_{i+1}(T_{n+1}) = Q_i(T_n) + (1 - l/f)\Delta X \tag{20a}$$

In the absence of leakage and after invoking the condition $Q(X = 0, T) = Q(X, T = 0) = 0$, (20a) simplifies to the form that replicates the kinematic wave solution:

$$Q_{i+1}(T_{n+1}) = Q_i(T_n) + \Delta X \tag{20b}$$

Despite the formal similarity of (6) and (9), for the problem at hand, Muskingum-Cunge routing schemes are for several reasons ill-suited for the computation of depth profiles. First, and foremost, the depth at $X = 0$ is unknown and so the integration cannot start there. But even if $H(0, T)$ is assumed, the explicit, downslope integration predicts free surfaces that plateau near $X = 1$; this is the consequence of ignoring the condition $H(X = 1, T) = 0$, which would tie the profile down at the outflow section. If, alternatively, integration started at $X = 1$, where $H = 0$, it would proceed upslope, contrary to the physics of gravity-dominated flow. As a simpler alternative to a standard solution for two-point boundary value problems [*Koussis and Lien*, 1982], we decided to solve for the depth taking advantage of the known discharge. Either (5) or (2) is appropriate; we chose (5) (nondimensional, (8)) because of its simplicity.

After study we discarded explicit algorithms that march the integration out from $X = 1$: (1) first-order-accurate derivative approximations in $[X_{i+1}, X_i]$ and evaluation of the algebraic terms at X_i yielded smooth but not very accurate results; (2) evaluation of the algebraic terms at X_{i+1} caused oscillations; (3) a scheme with a first-order-accurate "starter" for the first interval and a second-order-accurate, central finite difference approximation thereafter, also suffered from oscillations (large errors are incurred near $X = 1$, where $\partial H/\partial X$ is large, and are propagated upgradient, corrupting the solution). An implicit calculation of depth profiles, based on central finite differences and a tridiagonal system of equations, is reliable but tedious. We therefore chose to integrate (8) formally with respect to X,

$$H(X, T) = (F/H_o) \int_X^1 Q(\xi, T) \exp[(X - \xi)/H_o]\, d\xi \tag{21}$$

and to evaluate the integral numerically by the trapezoidal rule. This method proved as accurate as the implicit procedure but much simpler. In both approaches we used intervals $\delta X < \Delta X$, to capture marked variations near $X = 1$, interpolating the necessary discharge values between points ΔX apart.

In concluding this part, we wish to point out that (16), used with a variable θ, can approximate nonlinearity; θ is computed by (18a) with H_o replaced in each ΔX by an average $\langle H_o \rangle$ calculated as the three-point average at nodes (i, n), $(i, n + 1)$ and $(i + 1, n)$; each value may be estimated as $H \approx QF$ from appropriate flow rates. Depth profiles can be calculated similarly.

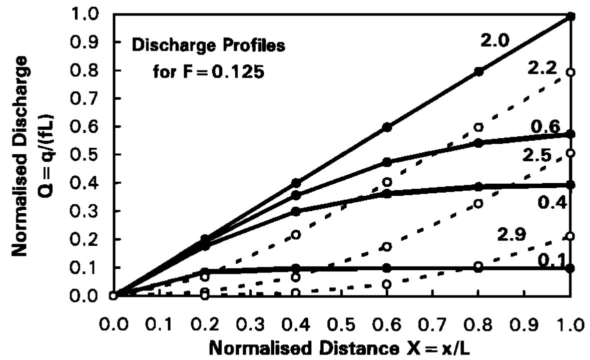

Figure 2. Evolution of discharge profiles over impervious barrier for $F = 0.125$, computed with $\Delta X = 0.2$; $\mathbf{P} \approx 2$. Normalized times are shown on build-up (solid lines) and recession (dashed lines) profiles.

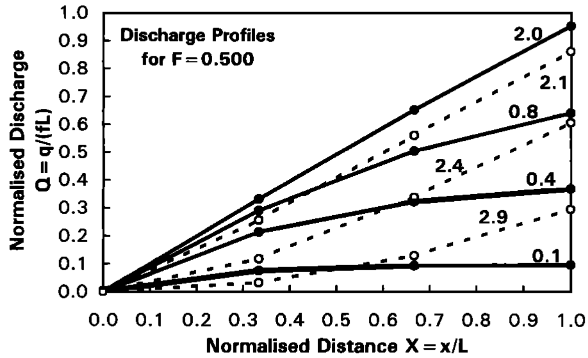

Figure 4. Evolution of discharge profiles over impervious barrier for $F = 0.5$, computed with $\Delta X = 1/3$; $\mathbf{P} \approx 1.5$. Normalized times are shown on build-up (solid lines) and recession (dashed lines) profiles.

5. Applications

5.1. Numerical Determination of Depth and Discharge Distributions

We compute subsurface storm flow generated by infiltration step inputs of normalized intensities $F = 0.125$ and $F = 0.5$ and duration 2, followed by a recession; for $F = 0.5$ the steady state depth at the hilltop is significantly larger than zero ($H(0) = 0.1$, compared to max $H \approx 0.3$ [*Koussis*, 1992]). The set of realistic conditions, slope $S = 0.2$ ($\varphi \approx 12°$), hydraulic conductivity $K = 1$ m/h, and infiltration rate $f = 0.01$ m/h, yields $F = 0.25$, indicating that the limit of validity of the kinematic wave model, $F = 0.175$ [*Beven*, 1981], is readily exceeded. Departures from nonlinearity are gauged by comparing our results with those of *Pi and Hjelmfeld* [1994].

Figure 2 shows the evolution of discharge profiles for $F = 0.125$, computed in five steps ($\Delta X = 0.2$; $\mathbf{P} \approx 2$); Figure 3 depicts corresponding depth profiles, computed with $\delta X = 0.05$. Figure 4 shows discharges for $F = 0.5$, computed with $\Delta X = 1/3$ ($\mathbf{P} \approx 1.5$) and Figure 5 depth profiles, computed with $\delta X = 1/21$; the sizeable depth at $X = 0$ is ignored by the kinematic wave theory. Outflows, computed in one and in multiple steps, are graphed in Figure 6 for $F = 0.125$ and in Figure 7 for $F = 0.5$; the kinematic wave solution is shown as a broken line. Discharge profiles are smooth, in contrast to the rapid variation of the depth near $X = 1$; the accuracy of the one-step calculation is noted. As expected, agreement with the nonlinear solution is closer for $F = 0.125$ than for $F = 0.5$, despite use of $H(X = 0, T) = 0$ by *Pi and Hjelmfeld* [1994] in the first case. Linear and nonlinear depth hydrographs are compared for $F = 0.125$ in Figures 8a and 8b.

For a test with leakage we use $F = 0.5$, for emphasis, and assume that cracks reduce the hydraulic conductivity contrast between drainage layer and landfill liner to $K/k = 10^3$. The data are as follows: drain spacing is 100 m ($L = 50$ m), slope $S = 0.1$, and liner thickness $b = 2$ m. With $H_o \approx F/2 = 0.25$, $h_o = H_o \cdot LS = 1.25$ m, and $h_o/b = 0.625$. By (13), leakage rate $l = 1.625k$, or $L_f = 1.625k/KS^2 = 0.1625$, and $F_{\text{net}} = F - L_f \approx 0.34$. $F = 0.34$ and $H_o \approx 0.17$ ($h_o = 0.85$ m) give $h_o/b = 0.425$; then $l = 1.425k$, $L_f = 0.1425$, and $F_{\text{net}} \approx 0.36$. Another iteration yields $F_{\text{net}} \approx 0.35$ and $L_f/F = 0.3$. For $K/k = 10^4$ a single calculation suffices, yielding $L_f = 0.01625$, $F_{\text{net}} \approx 0.48$, and $L_f/F = 0.03$. In summary, then,

$$L_f/F = l/f = k(1 + h_o/b)/f = k(1 + h_o/b)/KS^2F$$

$$\approx k(1 + LSF_{\text{net}}/2b)/KS^2F \tag{22a}$$

and

$$F_{\text{net}} = F - L_f \tag{22b}$$

The outflow for $F = 0.5$, with leakage from a barrier 1000 times less pervious than the drainage layer, is shown in Figure

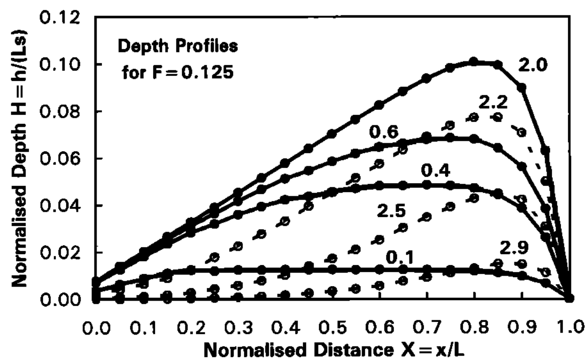

Figure 3. Evolution of depth profiles over impervious barrier for $F = 0.125$, computed with $\delta X = 0.05$. Normalized times are shown on build-up (solid lines) and recession (dashed lines) profiles.

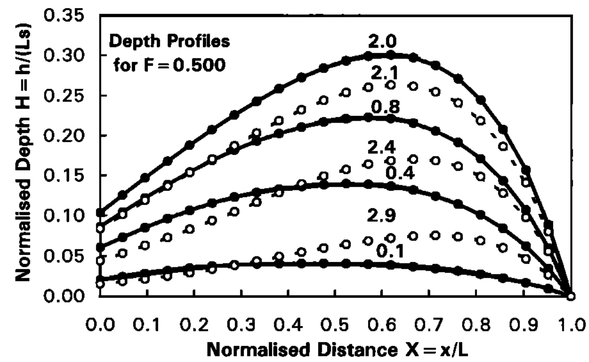

Figure 5. Evolution of depth profiles over impervious barrier for $F = 0.5$, computed with $\delta X = 1/21$. Normalized times are shown on build-up (solid lines) and drainage (dashed lines) profiles.

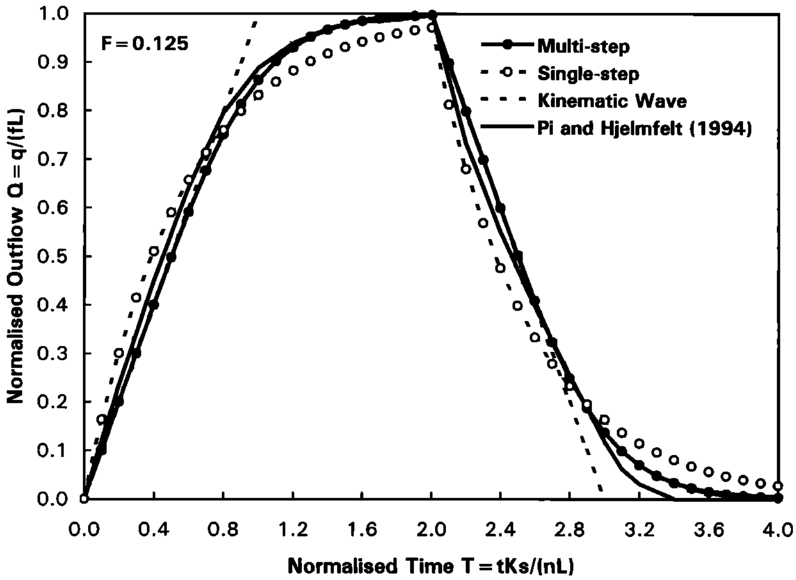

Figure 6. Comparison of outflow hydrographs for lateral drainage over an impervious barrier for $F = 0.125$, computed in multiple and in a single steps, with the nonlinear solution of *Pi and Hjelmfeld* [1994] and the kinematic wave solution (dashed line).

9 and compared to the no-leakage case; again, single- and multiple-step results are presented that demonstrate the utility of the one-step solution for design. With F reduced to $F_{\text{net}} = 0.35$, $\Delta X = 1/4$ is permissible ($\mathbf{P} \approx 1.5$). For drainage with leakage, portions of the bed may run dry during the recession; however, leakage would continue to be computed because of the linearization of the gradient-driven flow, resulting in negative solution values. To avoid this, discharges are maintained at zero once the bed dries.

We finally examine data from a laboratory drainage experiment ($T > 0$, $F = 0$) of *Sanford et al.* [1993]. In the particular experiment the bed was tilted and water was added, more near the top of the hill than its foot, while keeping the outflow section closed, until the depth was nearly uniform; the lower barrier was then removed suddenly, and the water supply stopped. This contrived constant initial depth setup is not realizable in nature. It can be achieved exactly in the laboratory by confining the saturated porous medium in a box, tilting it, and then removing the top and lower faces suddenly. For h = const, only gravity flow exists, and gradient-driven flow is zero. Total flow vanishes when the pressure gradient along the top counterbalances the bed slope, (1). The pressure is thus not atmospheric and the flow not unconfined, which makes us skeptical of the generality of the empirical drainage formulas of *Sanford et al.* [1993].

It is impossible to model this case rigorously; nevertheless

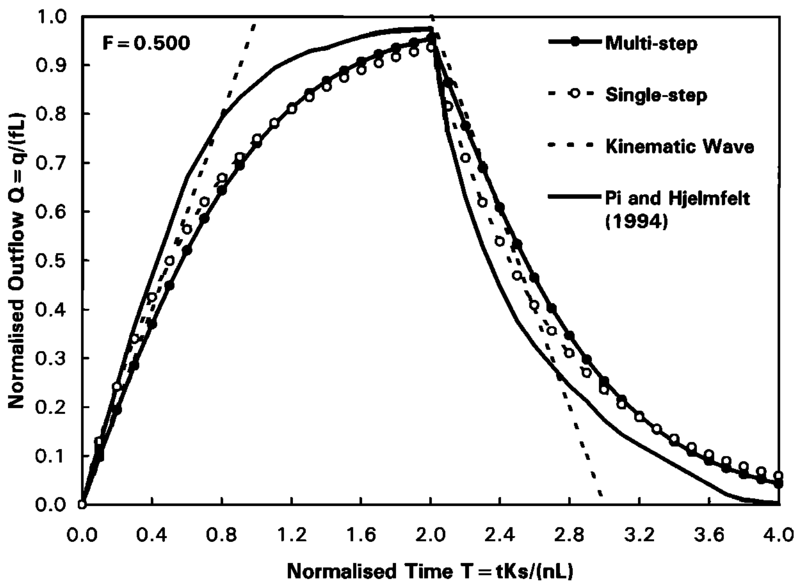

Figure 7. Comparison of outflow hydrographs for lateral drainage over an impervious barrier for $F = 0.5$, computed in multiple and in a single steps, with the nonlinear linear solution of *Pi and Hjelmfeld* [1994] and the kinematic wave solution (dashed line).

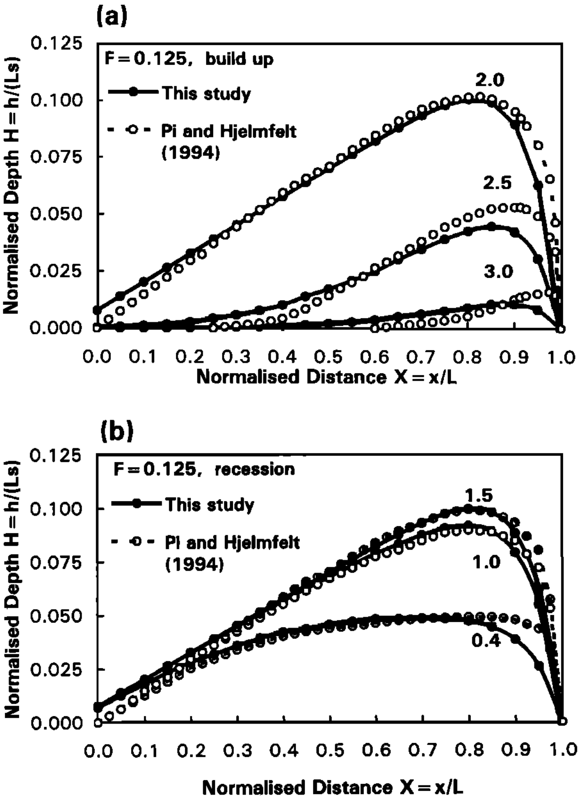

Figure 8. Comparison of depth hydrographs for lateral drainage over an impervious barrier for $F = 0.125$, (a) build-up and (b) recession phases, with the nonlinear linear solution of *Pi and Hjelmfeld* [1994].

we can examine the cumulative outflow evolution, an integral quantity less sensitive to initial conditions. The task was exacerbated by the lack of initial infiltration data. To advance the solution, we estimated an initial discharge profile by numerical differentiation of the first depth profile. The comparison indicates acceptable agreement, given the uncertain initial data. Early data exceed computed values because h = const causes faster early drainage than the model can anticipate, affecting flow more on low slopes, $S = 0.089$ (Figure 10a), than on high ones, $S = 0.139$ (Figure 10b).

5.2. Quasi-analytical Determination of the Outflow Rate

A one-step solution ($\Delta X = 1$) of the discharge is as convenient as an analytical formula for the outflow at the foot of the hill, $Q_{\text{out}}(T)$, and may be sufficiently accurate. It is derived from (16a) by recognizing that the flow at the hill top is zero and $C_1 + C_2 + C_3 = 1$, whence

$$Q_{\text{out}}(T_{n+1}) = C_3 Q_{\text{out}}(T_n) + (1 - C_3)(1 - l/f) \tag{23a}$$

Repeated application of (23a) over time leads to the convergent ($C_3 < 1$) geometric series

$$Q_{\text{out}}(T_{n+1}) = (1 - C_3)(1 - l/f)[1 + C_3 + C_3^2 + \cdots + C_3^n]$$

$$= (1 - C_3)(1 - l/f) \sum_{m=0}^{n} (C_3)^m \tag{23b}$$

The sum of the series is $(1 - C_3^n)/(1 - C_3)$, and the final, compact result is

$$Q_{\text{out}}(T_{n+1}) = (1 - l/f)(1 - C_3^n) \tag{24}$$

In the limit $T \to \infty$, (24) gives the correct steady state solution, $Q_{\text{out}}(T \to \infty) = 1 - l/f$. If the outflow at the time of cessation of infiltration, T^*, is Q^*_{out}, the drainage rate for times thereafter, indicated by $\tau = T - T^*$, can be computed simply as

$$Q_{\text{out}}(\tau_\nu) = (C_3)^\nu Q^*_{\text{out}} \tag{25}$$

From the perspective of practical work (24) and (25) may be the principal results of this work. They provide a simple and rational basis for (1) parameterizing the response function of a hill slope and (2) computing lateral flow in landfill drainage.

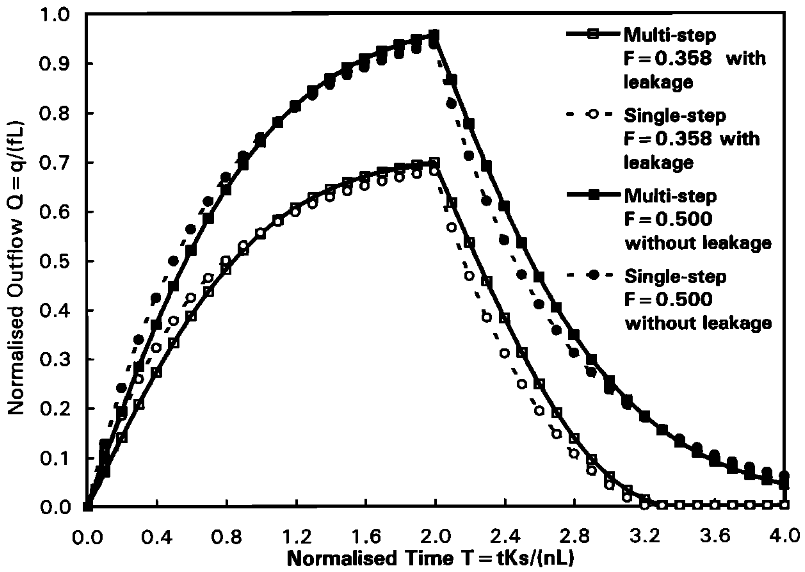

Figure 9. Comparison of multiple- and single-step solutions for the outflow hydrograph: solid symbols are for drainage over an impervious barrier for $F = 0.5$; open symbols are for drainage over a leaky barrier ($S = 0.1$, $b = 2$ m, $L = 50$ m, and $K/k = 1000$) for $F_{\text{net}} = F - L_f = 0.358$.

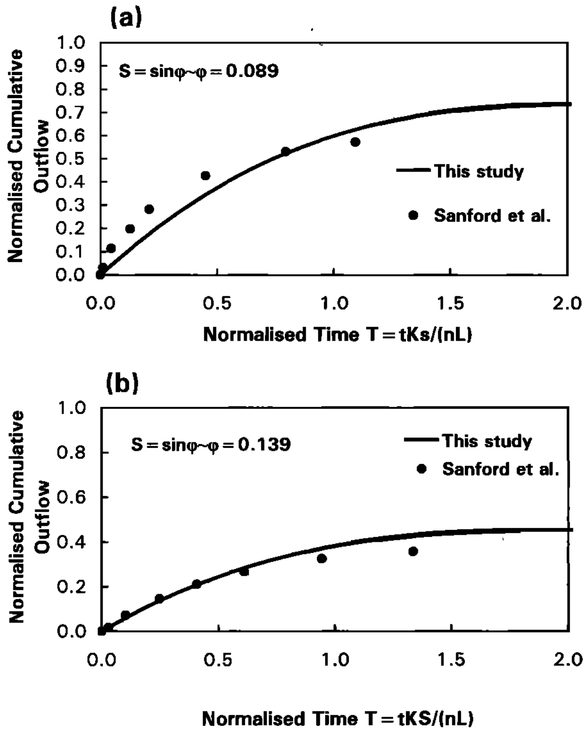

Figure 10. Comparison of computed cumulative outflow volume hydrographs with laboratory data (symbols) of *Sanford et al.* [1993] for two slopes: (a) $S = 0.089$ and (b) $S = 0.137$.

The parameters that control response time and hydraulic diffusion are more visible in the dimensional form of (24) and (25). In this context we note that (25) also predicts the exponential decline of the outflow included in the analytical solutions [*Brutsaert*, 1994; *Chapman*, 1995], when the exponential Muskingum scheme [*Koussis*, 1980] is used. In that scheme's one-step format, $C_3 = \exp[-\Delta T/(1-\theta)] = \exp[-\Delta\tau/(1-\theta)]$ and (25) gives the response of a distributed reservoir,

$$Q_{\text{out}}(\tau_\nu) = Q^*_{\text{out}} \exp[-\nu\Delta\tau/(1-\theta)] = Q^*_{\text{out}} \exp[-\tau_\nu/(1-\theta)] \quad (26)$$

6. Summary and Conclusions

We have presented a method for computing subsurface storm flow from hill slopes and lateral drainage flow in landfills. The method is based on the linear, extended Boussinesq equation for unconfined flow over a sloping, leaky base and accounts for the dominant gravity flow and for the gradient-driven flow, which the kinematic wave equation neglects. The boundary conditions for depth and discharge at the drain have been discussed and it has been shown that zero depth at the drain yields also a reasonable boundary condition for the discharge. For the case of a leaky base, or liner, a procedure has been developed for estimating net infiltration from gross infiltration and from properties of the base (liner) that control leakage. The evolution of discharge is computed first by using the Muskingum-Cunge routing scheme; depths are determined, if needed, by integrating Darcy's equation for the known discharges. This is a parsimonious methodology, which is also accurate and robust with proper grid design. The application of the proposed methodology is demonstrated in examples of buildup and recession phases, with and without leakage. The outflow over the sloping barrier into a drain or stream at the foot of the hill slope can also be computed in a single space step with adequate accuracy for practical applications. Such a procedure is as convenient as an analytical solution and can be used as a basis for the parameterization of the hydrologic response function of a hill slope and for the computation of certain aspects of landfill drainage hydraulics.

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