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Response of sloping unconfined aquifer to stage changes in adjacent stream. I. Theoretical analysis and derivation of system response functions

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Summary We study the interaction of a stream with a sloping unconfined aquifer that the stream is assumed to fully penetrate. The analysis applies to flow in a vertical section, considers the existence of a low-conductivity streambed layer and the flow in the aquifer to be induced by variations of the stream stage. Invoking the Dupuit assumption yields the 1-D Boussinesq equation, extended for a sloping base. The Boussinesq equation is linearised, the derived flow model is critiqued and an objective procedure for determining the linearisation level is developed. We solve the linear governing equation by the method of Laplace transform, with analytical inversion; the horizontal-aquifer case is treated in the zero-slope limit. The system response function is derived for the general case (sloping aquifer, sediment bed layer) and for several specific cases, and solutions are verified against known analytical results. Responses are contrasted for aquifers on positive, negative and zero slopes to step changes in the stage of streams with and without a sediment bed layer. The solutions give the aquifer stage and flow rate, the flow exchange rate at the stream–aquifer interface and the exchanged water volumes (bank storage/release).

Introduction

The interaction of a stream with an adjacent aquifer interests hydro-scientists/engineers because it occurs in a variety of cases, such as the conjunctive management of surface and ground water resources, stream base flow and the modification of a flood wave through the exchange of water across the stream banks (Sophocleous, 2002). The disparate characteristic response times of aquifer and stream flow make the computation of the modification of streamflow through its interaction with an adjacent aquifer a significant numerical challenge (Perkins and Koussis, 1996). In this work we derive the aquifer response by analyt-

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Notation

Definition of symbols with dimension (L, length; T, time)

a	solution parameter entailing the ratio of velocity to diffusion coefficient [L^{-1}]
b	solution parameter [L^{-1}]
b_s	thickness of low conductivity sediment layer in streambed [L]
c_1, c_2	coefficients in the solution that are determined through the boundary conditions [LT]
D	diffusion coefficient [$L^2\,T^{-1}$]
D_o	diffusion coefficient, dimensionless [–]
$f(\cdots)$	function of argument in ( )
F	Laplace-transformed normalised depth of aquifer [–]
h	height of water column measured normal to the bed, depth of aquifer [L]
h_o	depth of linearisation of the aquifer [L]
H	normalised depth of aquifer [–]
H_o	normalised linearisation depth of the aquifer [–]
i	imaginary unit $(-1)^{1/2}$
K	hydraulic conductivity of aquifer [$L\,T^{-1}$]
K_s	hydraulic conductivity of sediment streambed layer [$L\,T^{-1}$]
l	streambed leakance [L]
L	length of aquifer [L]
n	drainable porosity (specific yield) of aquifer [–]
q	aquifer discharge per unit width [$L^2\,T^{-1}$]
Q	aquifer discharge per unit width normalised by KLS^2 (sloping) or KL (horizontal) [–]
R_ν	νth residual used in the development of the solution [L]
s	Laplace transform variable [T^{-1}]
s_ν	poles in the power series development of analytic functions (theorem of residua) [T^{-1}]
S	slope of aquifer basis, defined as $\sin\varphi$ [–]
t	time [T]
T	normalised time [–]
u	system response function, dimensional [T^{-1}]
U	system response function, dimensionless [–]
vol	volume of water exchanged with aquifer, per unit stream length [L^2]
V	kinematic wave linear pore velocity of aquifer, KS/n [$L\,T^{-1}$]
VOL	normalised volume of water, per unit stream length, exchanged with aquifer [–]
x	distance measured along the aquifer base [L]
X	normalised distance measured along the aquifer base [–]
y	depth at stream–aquifer interface [L]
Y	dimensionless depth at stream–aquifer interface [–]
z_ν	νth root of the equation $\tan z = f(z, a, l, L)$ [–]
η_ν	function of parameters in expression for the νth residuum, η_ν (a, l, L, z_ν, y) [–]
ζ	elevation above a datum [L]
$\xi_\nu(x)$	spatial function in expression for the νth residuum [–]
τ	dummy variable of integration over time [T]
φ	inclination angle of aquifer base against the horizontal [–]
Φ	hydraulic potential [L]
ψ	dummy variable of integration [–]
Ψ	argument of error function [–]

ical means, prescribing the variation of the stream stage as an aquifer boundary condition.

Barlow and Moench (1998) review an array of Laplace-transform-based solutions for the interaction between confined, leaky and unconfined aquifers with an adjacent stream, which the paper of Moench and Barlow (2000) summarises; they also document two computer codes for solving the interaction problem through numerical convolution. In the relevant literature the aquifer base is taken as horizontal, and, as a consequence, upon linearisation, the equation governing planar flow in an unconfined aquifers under the Dupuit approximation (Boussinesq equation) is of the pure diffusion type. In their pioneering work, Cooper and Rorabough (1963) solve that equation, deriving analytical expressions for the flow that develops in a horizontal aquifer due to a wave-like variation of the stream stage. We extend that work here to include the more realistic case of a stream interacting with an unconfined aquifer on a sloping base. Thus, after linearisation of the extended Boussinesq equation, we obtain a linear advection–diffusion-type equation that we solve by the Laplace transform method.

The structure of this first in a series of two papers is as follows. The mathematical formulation of the physics is presented first, with discussion, and also in non-dimensional form. The analytical solution methodology is detailed then, treating the case of a step change of the stream stage, from which the system response function (SRF) is derived; the solution for a horizontal aquifer is obtained in the zero-slope limit. The solutions give the aquifer stage and flow rate, the flow exchange rate at the stream-bank and the volumes exchanged (bank storage). The aquifer responses for horizontal and for positive or negative sloping bases are contrasted for step changes and for a streambed with and without a sediment layer. The added complexity of recharge is ignored mainly due to its minor influence relative to the stream.

The mathematical representation of the physics

The hydraulic description of flow in an unconfined aquifer (theory of Dupuit) of uniform hydraulic conductivity K and drainable porosity n (specific yield) approximates the pressure in the water column of height h normal to the bed as hydrostatic. The flow potential Φ is then the sum of the elevation at a point in the fluid, ζ, and the hydrostatic head there, $h\cos\varphi$. The gradient of elevation in the flow direction x, measured from the stream–aquifer interface along the base that is inclined at the angle φ against the horizontal, is $d\zeta/dx = S = \sin\varphi$. By Darcy's law, the discharge per unit width through the aquifer shown in Fig. 1 (planar flow) is

$$q(x,t) = -h\left(K\frac{\partial\Phi}{\partial x}\right) = -Kh\left(S + \frac{\partial h}{\partial x}\cos\varphi\right). \tag{1}$$

Combining Eq. (1) with the storage balance (mass conservation stated as volume balance)

$$n\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{2}$$

yields the equation of Boussinesq (1877) for unconfined flow in a porous layer on an inclined base (Henderson and Wooding, 1964; for a thorough analysis, see Wooding and Chapman, 1966 and Childs, 1971):

$$n\frac{\partial h}{\partial t} - KS\frac{\partial h}{\partial x} - \cos\varphi K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) = 0. \tag{3}$$

General analytical solutions are not known for the non-linear Eq. (3), only solutions for special flows in horizontal aquifers. The kinematic wave approximation [indicated already by Boussinesq (1877)] offers a ready simplification for $S \gg |\partial h/\partial x|$, whence $q \approx hKS$, is obtained by neglecting the term $\cos\varphi K\partial(h\partial h/\partial x)/\partial x$ in Eq. (3), but fails as $\varphi \to 0$. Judicious linearisation around some depth h_o renders the governing equations analytically tractable:

$$q = -hKS - h_oK\cos\varphi\frac{\partial h}{\partial x} \tag{4}$$

$$n\frac{\partial h}{\partial t} - KS\frac{\partial h}{\partial x} = Kh_o\cos\varphi\frac{\partial^2 h}{\partial x^2}. \tag{5}$$

The advection–diffusion equation (5) (velocity $-KS/n$, diffusion coefficient $K\,h_o\cos\varphi/n$) is the linear advection–dispersion model of Koussis and Lien (1982) for flow in a sloping porous layer that has been used in numerous studies (e.g., Koussis, 1992; Brutsaert, 1994; Koussis et al., 1998; Verhoest and Troch, 2000; Pauwels et al., 2002; Akylas et al., 2006).

Introducing the normalised variables for the sloping aquifer

$$X = x/L; \quad H = h/(LS); \quad T = tKS/(nL); \quad Q = q/(KLS^2), \tag{6a}$$

the non-dimensional equation governing the depth becomes

$$\frac{\partial H}{\partial T} - \frac{\partial H}{\partial X} = H_o\cos\varphi\frac{\partial^2 H}{\partial X^2} \tag{7a}$$

and the corresponding non-dimensional form of the discharge Eq. (4) reads

$$Q = -H - H_o\cos\varphi\partial H/\partial X. \tag{8a}$$

The slope term vanishes in the case of a horizontal aquifer and $q = -Kh_o\partial h/\partial x$. The governing equation (7a) reverts then to a diffusion-type, in terms of re-defined normalised variables:

$$X = x/L; \quad H = h/L; \quad T = tK/(nL); \quad Q = q/(KL). \tag{6b}$$

$$\frac{\partial H}{\partial T} = H_o\frac{\partial^2 H}{\partial X^2}, \tag{7b}$$

$$Q = -H_o\partial H/\partial X. \tag{8b}$$

By replacing h_o by B = const. and n by the specific storage S_s, Eqs. (7b) and (8b) describe flow in a constant-thickness confined aquifer. Henceforth we adopt the compact notation $D_o = H_o\cos\varphi$.

The flow vanishes at the inland boundary, $x = L$, $X = 1$, i.e., $q(1, t) = Q(1, T) = 0$; the water is thus assumed to al-

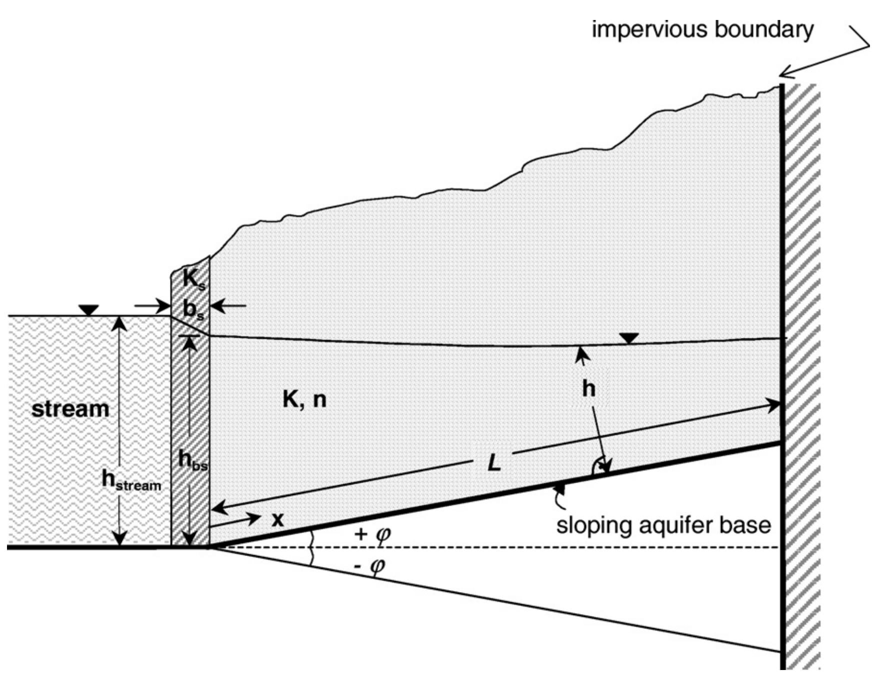

Figure 1 Cross-sectional schematic of stream and adjacent sloping aquifer, with definition of symbols.

ways be at that boundary (no moving boundary on a slope is considered). This condition translates for the sloping and for the horizontal aquifer, respectively, to

$$\partial h/\partial x = -Sh(L,t)/h_o \cos\phi, \quad \text{or}$$
$$\partial H/\partial X = -H(1,T)/H_o \cos\phi = -H(1,T)/D_o \tag{9a}$$
$$\partial h/\partial x = 0, \quad \text{or} \quad \partial H/\partial X = 0. \tag{9b}$$

The stream–aquifer boundary is simplified as vertical and the stream is assumed to penetrate the aquifer fully; due to the Dupuit–Forchheimer approximation, a seepage face is ignored and the water level is continuous across the stream–aquifer interface. A streambed layer of low conductivity is taken into account. We consider a step change in stream stage from 0 to y (0 to Y non-dimensionally), relative to the initial stream level. The actual initial and final values affect only the appropriate linearisation level.

The condition at the stream–aquifer interface depends on the presence or not of a sediment streambed layer. In the absence of such a layer, the depth at the interface boundary, above an initial stream depth h_{is}, writes

$$\text{step rise}: h(0,t=0_-) = 0, h(0,t>0_+) = y, \quad \text{or}$$
$$H(0,T=0_-) = 0, H(0,T>0_+) = Y \tag{10a}$$
$$\text{step drop}: h(0,t=0_-) = y, h(0,t>0_+) = 0, \quad \text{or}$$
$$H(0,T=0) = Y, H(0,T>0_+) = 0 \tag{10b}$$

If a bed layer of lower conductivity K_s is present, a linear depth profile is prescribed across its thickness b_s. Flow continuity at the streambed–aquifer interface requires then (ignoring the ill-defined bed slope across the schematised interface)

$$K_s \frac{h_{\text{stream}} - h_{b_s}}{b_s} = -K \left.\frac{\partial h}{\partial x}\right|_{b_s}. \tag{11}$$

The initial condition for the depth follows from Eq. (4) for $q(x, 0) = Q(X, 0) = 0$, which gives

$$dh/dx = -Sh/(h_o \cos\varphi), \tag{12}$$

or equivalently from the steady-state solution of Eq. (7) and for a known initial depth h_{is} at the stream aquifer interface, for an appropriate boundary:

$$h(x,0) = h_s(x) = h_{is} \exp(-Sx/h_o \cos\varphi), \quad \text{or}$$
$$H(X,0) = H_s(X) = H_{is} \exp(-X/D_o), \quad \text{and} \tag{13a}$$
$$h(x,0) = h_s(x) = h_{is}, \quad \text{or}$$
$$H(X,0) = H_s(X) = H_{is} \text{ for a horizontal aquifer}. \tag{13b}$$

We analyse the linearisation shortcomings because we found no relevant discussion in the literature. The exponential profile Eq. (13a), which holds for a stagnant aquifer, is counter-intuitive and an artefact of linearising Eqs. (1)–(4); e.g., a flawed consequence of Eq. (4) is that, if $dh/dx \neq 0$, the flow does not vanish when $h = 0$. In addition, Eq. (13a) yields $h_s(L) > 0$ for any h_{is} and L, i.e., the profile climbs the slope indefinitely, and $dh/dx \neq -\tan\varphi$, at L. These results conflict with the nonlinear flow Eq. (1), which gives for $q = 0$ (besides the trivial $h = 0$) the horizontal profile $dh/dx = -\tan\varphi$, with reach limited to $h_{is}/\sin\varphi$ and hydraulic potential $\Phi = h_{is} = const.$ (datum at the river bed), since

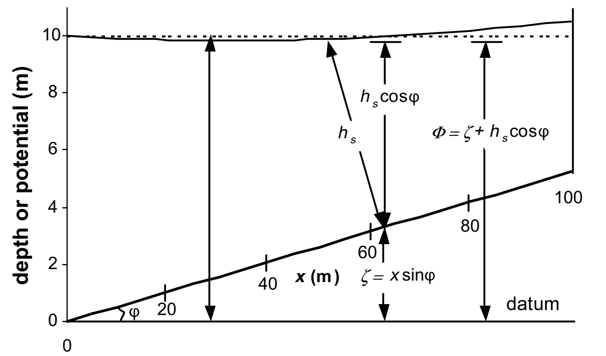

Figure 2 Comparison of steady flow profiles of depth and hydraulic potential between linear and nonlinear solutions.

$$h_{is} = \zeta + h\cos\varphi = x\sin\varphi + h\cos\varphi. \tag{14}$$

In contrast, profile Eq. (13a), with $dh/dx = -\tan\varphi h/h_o$, gives the variable hydraulic potential

$$\Phi = x\sin\varphi + h_{is}\cos\varphi \exp\left(-\frac{x\sin\varphi}{h_o\cos\varphi}\right) \neq h_{is}. \tag{15}$$

We optimise the linearisation level h_o of the steady flow profile by setting the mean *defect* of $(h_{is} - \Phi)$ (its integral over L) zero, which yields the implicit equation for D_o (or for h_o)

$$D_o = \left(1 - \frac{1}{2H_{is}}\right)(1 - e^{-1/D_o})^{-1}(\cos\varphi)^{-1}. \tag{16}$$

This equation is solved by iteration, starting from the horizontal profile estimate $D_o = (H_{is} - 0.5)\cos\varphi$. The steady-state depth and hydraulic potential profiles shown in Fig. 2 for $h_{is} = 10$ m, $L = 100$ m, $\varphi = 3°$ and $h_o\cos\varphi \approx 8.2$ m exemplify the linear solution's approximation of the horizontal surface. That Φ ($x \geqslant 64$ m) > h_{is} is a consequence of optimising $h_o\cos\varphi$ to give $\Phi = h_{is}$ on the profile-average. These flaws are minimised in field applications by model calibration, as demonstrated in the sequel paper Koussis et al. (in review). The linear model is free of the noted flaws, if the aquifer is horizontal.

Because Eq. (16) holds for steady flow, we re-visit the issue of optimal linearisation in ''Sensitivity study'' for transient flow. Yet, it seems advisable to assess the appropriateness of the linear model on steep slopes, as a large range of variation reduces the accuracy of any linearisation; however, on large slopes, the linear gravity term will be dominant, making the linearised diffusion term a secondary correction.

Development of analytical solution

General methodology

The solution that we develop is for a step change y of the stream stage relative to an initial stream depth h_{is} and the corresponding steady-state profile in the aquifer. That this is equivalent to solving for a step change y and zero-depth initial condition can be seen by considering the solution to consist of a steady state and a transient part. Then, the steady-state solution due to the initial stream depth h_{is} satisfies the steady state form of Eq. (5), leaving the transient solution, due to y, to be satisfied by Eq. (5), subject to a zero initial condition.

Laplace-transformation (variable s) of Eq. (5) yields the differential equation in space for the transformed depth variable F, which incorporates a zero-depth initial condition,

$$\frac{Kh_o \cos\varphi}{n}\frac{d^2F}{dx^2} + \frac{KS}{n}\frac{dF}{dx} - sF = 0, \tag{17a}$$

or restated, using the linear pore velocity $V = KS/n$ and the diffusivity $D = Kh_o \cos\varphi/n$,

$$D\frac{d^2F}{dx^2} + V\frac{dF}{dx} - sF = 0. \tag{17b}$$

The condition applied at the stream–aquifer interface depends on the physical situation. When a bed layer of lower conductivity sediments is present, Eq. (11) applies; upon transformation and after placing the origin of x-axis at the beginning of the aquifer (i.e., the boundary condition applies at the aquifer-side edge of the streambed layer), Eq. (11) yields

$$F(x=0) = y/s + (K/K_s)b_s dF/dx|_{x=0} = y/s + l dF/dx|_{x=0}; \tag{18a}$$

the scaled bed layer thickness $(K/K_s)b_s = l$ is called the streambed *leakance*, which is a key parameter of the stream–aquifer interaction (Sophocleous et al., 1995). Substituting for that layer an equivalent aquifer length l that offers the same flow resistance as the streambed layer is approximately correct for steady flow, but not for transient flow, because the storages of l and b_s differ and mass conservation is violated. In the absence of a streambed layer $b_s = 0$ and

$$F(x=0) = y/s. \tag{18b}$$

The no-flow condition at the landside boundary of the aquifer $x = L$,

$$dF/dx|_{x=L} = -VF(x=L)/D, \tag{19a}$$

simplifies in the case of a horizontal aquifer ($S = 0$ and thus $V = 0$) to

$$dF/dx|_{x=L} = 0. \tag{19b}$$

With the abbreviations $a = -V/2D$, $b = (a^2 + s/D)^{1/2}$, the general solution of Eq. (17) writes

$$F = c_1 \exp[(a+b)x] + c_2 \exp[(a-b)x]. \tag{20}$$

The constants c_1, c_2 are determined from the boundary conditions; Eq. (18a) is satisfied for

$$c_2 = -\frac{y/s + c_1[l(a+b)-1]}{l(a-b)-1}. \tag{21a}$$

In the absence of a streambed sediment layer, $l = 0$, Eq. (18b) applies and Eq. (21a) simplifies to

$$c_2 = y/s - c_1. \tag{21b}$$

Then, introducing Eq. (21a) into Eq. (20), we obtain

$$F(x) = c_1\left\{\frac{[l(a-b)-1]\exp[(a+b)x] - [l(a+b)-1]\exp[(a-b)x]}{[l(a-b)-1]}\right\} - \frac{y\exp[(a-b)x]}{s[l(a-b)-1]}. \tag{22}$$

Finally, the solution Eq. (22) satisfies the no-flow boundary condition at $x = L$ when

$$c_1 = \frac{y(a+b)\exp(-bL)}{2s[(la^2 + lb^2 - a)\sinh(bL) + (b - 2lab)\cosh(bL)]}, \tag{23a}$$

or, again, in the absence of a streambed sediment layer, when

$$c_1 = \frac{y(a+b)e^{-bL}}{2s[b\cosh(bL) - a\sinh(bL)]}. \tag{23b}$$

The general solution evaluates thus to

$$F(x) = \frac{ye^{ax}[a\sinh(bx - bL) + b\cosh(bx - bL)]}{s[(la^2 + lb^2 - a)\sinh(bL) + (b - 2lab)\cosh(bL)]}, \tag{24a}$$

specialising in the absence of a streambed sediment layer ($l = 0$) to

$$F(x) = \frac{ye^{ax}[b\cosh(bL - bx) - a\sinh(bL - bx)]}{s[b\cosh(bL) - a\sinh(bL)]}. \tag{24b}$$

Inverting from the s-domain to the time domain involves a complex formula entailing the evaluation of an integral, for which the theorem of residua (development of analytic functions in power series) is applied (Heinhold, 1948; Bronstein and Semendjajew, 1964). Following Brutsaert (1994), the inverse transform is given by the sum of the residua R_ν of $F(s)e^{st}$ at its ν singular points, or poles, s_o,

$$h(x,t) = \sum_\nu R_\nu. \tag{25}$$

The R_ν are defined as coefficients of $(s - s_o)^{-1}$ in the expansion of $F(s)e^{st}$ in the neighbourhood of the ν poles s_o. For poles of the order 1, a residuum is calculated by

$$R_\nu = \lim_{s\to s_o}[(s - s_o)F(s)e^{st}], \tag{26a}$$

or, if we set $F(s) = P(s)/T(s)$, by the equivalent form

$$R_\nu = \lim_{s\to s_o}\frac{P(s)e^{st}}{[T(s) - T(s_o)]/(s - s_o)} = \frac{P(s_o)e^{s_o t}}{T'(s_o)}, \tag{26b}$$

where $T' = dT/ds$. Eq. (26) has first-order poles at $s = 0$ and at the roots $s_\nu = s_o$ of

$$T'(s_\nu) = [la^2 + lb^2(s_\nu) - a]\sinh[b(s_\nu)L] + [b - 2lab(s_\nu)] \times \cosh[b(s_\nu)L]. \tag{27}$$

Letting $b(s_\nu)L = iz_\nu$ [$i = (-1)^{1/2}$ is the imaginary unit], $T'(s) = 0$ gives

$$\tan z_\nu = \frac{z_\nu(L - 2laL)}{lz_\nu^2 - la^2L^2 + aL^2}, \tag{28a}$$

which has an infinity of roots, $z_1, z_2, \ldots, z_\nu$. As a result, poles occur at $s = 0$ and at

$$s_\nu = -D\left(\frac{z_\nu^2}{L^2} + \frac{V^2}{4D^2}\right) = -\frac{D}{L^2}z_\nu^2 - Da^2, \quad \nu = 1, 2, 3, \ldots \tag{29a}$$

An exception to the above rule applies when the slope is negative, i.e. $a > 0$, and $0 < (L - 2laL)/(a\ L^2 - la^2L^2) < 1$. Only in that case Eq. (28a) does not have a solution in $0 \leqslant z_1 \leqslant \pi$ and the first root of $T'(s_\nu) = 0$ is found for real values of bL, whence the root z_1 is given by

$$\tanh z_1 = \frac{z_1(2laL - L)}{lz^2 + la^2L^2 - aL^2} \tag{28b}$$

and the corresponding pole becomes

$$s_1 = D\left(\frac{z_1^2}{L^2} - \frac{V^2}{4D^2}\right) = \frac{D}{L^2}z_1^2 - Da^2. \tag{29b}$$

The first term in Eq. (25) corresponds to the pole at $s = 0$, is calculated by Eq. (26a), gives the steady-state solution (relative to the initial steady flow) and is physically meaningful only when $1 - 2la > 0$,

$$R_o = y(1 - 2la)^{-1}e^{2ax}. \tag{30}$$

Eq. (26b) gives the remaining residua R_ν, $\nu = 1, 2, \ldots, \infty$, with $s_o = s_\nu$ from Eq. (27) for each z_ν:

$$R_\nu = \frac{-2ye^{ax}z_\nu\left[a\sin(z_\nu x/L - z_\nu) + \frac{z_\nu}{L}\cos(z_\nu x/L - z_\nu)\right]e^{s_\nu t}}{[a^2 + (z_\nu^2/L^2)][\sin z_\nu(2lz_\nu + Lz_\nu - 2laLz_\nu) + \cos z_\nu(lz_\nu^2 + aL^2 + 2laL - la^2L^2 - L)]}; \tag{31a}$$

when the root for bL is real [Eq. (28b), s_1 from Eq. (29b)], R_1 is calculated by

$$R_1 = \frac{-2ye^{ax}z_1\left[a\sinh(z_1 x/L - z_1) + \frac{z_1}{L}\cosh(z_1 x/L - z_1)\right]e^{s_1 t}}{[a^2 - (z_1^2/L^2)][\sinh z_1(2lz_1 + Lz_1 - 2laLz_1) + \cosh z_1(lz_1^2 - aL^2 - 2laL + la^2L^2 + L)]}. \tag{31b}$$

Composing solutions of Eq. (5) for a step input is now straightforward: the $h(x, t)$-series of Eq. (25) are formed from Eqs. (30)–(31), which are evaluated with Eqs. (28)–(29). The SRF, $u(x, t)$, is the solution for a unit-impulse input (determined by subtracting two unit steps, t_{pulse} apart, and taking the limit $t_{\text{pulse}} \to 0$) that is obtained directly from

$$u(x,t) = \left.\frac{\partial h(x,t)}{\partial t}\right|_{y=1}. \tag{32a}$$

With the notation in Eq. (31), $R_\nu = \eta_\nu(a,l,\ L,z_\nu,y)\xi_\nu(x) \times \exp(s_\nu t)$, Eq. (32a) is restated as

$$u(x,t) = \sum_{\nu=1}^{\infty} s_\nu R_\nu(x,t) = \sum_{\nu=1}^{\infty} \eta_\nu(a,l,L,z_\nu,y=1)s_\nu\xi_\nu(x)e^{s_\nu t}. \tag{32b}$$

Our purpose for deriving the SRF is to use it in the solution of more complex problems, in which the forcing varies gradually in time, through application of the convolution principle. This will be shown in the companion paper Koussis et al. (in review).

Special cases of the general solution

In the following we present some special cases such as the solution for a sloping aquifer in the absence of a streambed sediment layer, $l = 0$. In this case Eq. (31a) simplifies to

$$R_\nu = \frac{-2ye^{ax}z_\nu e^{s_\nu t}\sin\left(\frac{z_\nu x}{L}\right)}{z_\nu^2 + a^2L^2 - aL} \tag{33a}$$

and the z_ν are calculated from $\tan z_\nu = z_\nu/aL$. Additionally, in the event of a real bL-root [from Eq. (28b), specialised to $\tanh\ z_1 = z_1/aL$, and s_1 from Eq. (29b)], R_1 is calculated by

$$R_1 = \frac{-2ye^{ax}z_1 e^{s_1 t}\sinh\left(\frac{z_1 x}{L}\right)}{z_1^2 - a^2L^2 + aL}, \tag{33b}$$

which is meaningful only when $aL > 1$ [see text after Eq. (29a)], corresponding to values of $h_o < L\tan\varphi/2$. The solution is obtained by substituting Eqs. (30) and (33) into Eq. (25):

$$h(x,t) = y\left[e^{2ax} + e^{ax}\sum_{\nu=1}^{\infty}\frac{2z_\nu\sin\left(\frac{z_\nu x}{L}\right)}{aL - a^2L^2 - z_\nu^2}e^{-D\left(\frac{z_\nu^2}{L^2}+\frac{V^2}{4D^2}\right)t}\right], \tag{34}$$

The first term is the steady-state and the second the transient solution; for $t = 0$, the series sum is $-e^{ax}$, giving $h(x, 0) = 0$. Then, introducing Eq. (34) in Eq. (4) yields for the flow rate:

$$q(x,t) = -yKSe^{ax} \times \sum_{\nu=1}^{\infty}\frac{\frac{2z_\nu^2 h_o\cos\varphi}{LS}\cos\left(\frac{z_\nu x}{L}\right) + z_\nu\sin\left(\frac{z_\nu x}{L}\right)}{aL - a^2L^2 - z_\nu^2}e^{-D\left(\frac{z_\nu^2}{L^2}+\frac{V^2}{4D^2}\right)t}. \tag{35}$$

We are also in a position to give the solutions in dimensionless form (see Eq. (6a)):

$$H(X,T) = Y\left[e^{-X/D_o} + 2e^{-X/2D_o}\sum_{\nu=1}^{\infty}\frac{-z_\nu\sin(z_\nu X)}{1/2D_o + 1/4D_o^2 + z_\nu^2}e^{-(z_\nu^2+1/4D_o^2)D_oT}\right], \tag{36}$$

$$Q(X,T) = Ye^{-X/2D_o}\sum_{\nu=1}^{\infty}\frac{2z_\nu^2D_o\cos(z_\nu X) + z_\nu\sin(z_\nu X)}{1/2D_o + 1/4D_o^2 + z_\nu^2}e^{-(z_\nu^2+1/4D_o^2)D_oT}. \tag{37}$$

The SRF for a sloping aquifer, in the absence of a streambed sediment layer, is

$$u(x,t) = 2\frac{D}{L^2}e^{ax}\sum_{\nu=1}^{\infty}\frac{z_\nu\sin\left(\frac{z_\nu x}{L}\right)}{1 - \frac{aL}{a^2L^2+z_\nu^2}}e^{-\frac{D}{L^2}(z_\nu^2+a^2L^2)t}, \tag{38}$$

or in dimensionless form

$$U(X,T) = \frac{u(x,t)}{KS/nL} = 2D_oe^{-\frac{X}{2D_o}}\sum_{\nu=1}^{\infty}\frac{z_\nu\sin(z_\nu X)}{1 - \frac{aL}{a^2L^2+z_\nu^2}}e^{-(z_\nu^2+a^2L^2)D_oT}. \tag{39}$$

The flow rate between stream and aquifer $q(0, t)$ or $Q(0, T)$ is positive as aquifer inflow. This flow becomes infinite for $t = T = 0$, consistent with the discontinuity due to the step stage rise at the stream–aquifer interface, which leads to an infinite hydraulic gradient there. Storage per unit stream bank length is obtained from that flow rate through integration over time; it gives the volume of stream water that has entered/exited a unit width of the aquifer till time t:

$$\mathrm{vol}(t) = \int_0^t q(0,\tau)\mathrm{d}\tau, \tag{40}$$

$$\mathrm{vol}(t) = -2nyL\sum_{\nu=1}^{\infty} \frac{z_\nu^2}{(z_\nu^2 + a^2L^2)(aL - a^2L^2 - z_\nu^2)}\left[1 - \mathrm{e}^{-\frac{D}{L^2}(z_\nu^2 + a^2L^2)t}\right], \tag{41}$$

or, normalising with the water volume nyL,

$$\mathrm{VOL}(T) = \frac{\mathrm{vol}(t)}{nyL} = -2\sum_{\nu=1}^{\infty} \frac{z_\nu^2}{(z_\nu^2 + a^2L^2)(aL - a^2L^2 - z_\nu^2)}\left[1 - \mathrm{e}^{-(z_\nu^2 + 1/4D_0^2)D_oT}\right]. \tag{42}$$

It is noted that the convolution principle concerns the response relative to the initial condition. Convolution is straightforward for a horizontal aquifer, as we can absorb any constant initial condition in a net variable ($h - h_{is}$), making the new initial depth zero. In the case of a sloping aquifer however, the initial depth is variable; for this reason the solution has been developed relative to that space-variable initial condition (see text at the beginning of this section). We reiterate that for the derived solution to be valid, the water must always reach the inland boundary. Next, we apply superposition in order to obtain the response of a sloping aquifer to an abrupt change Δy (positive or negative) from an initial stream stage h_{is}. The solution, stated including the initial condition, reads

$$h(x,t) = h_{is}\mathrm{e}^{2ax} + \Delta y\left(\mathrm{e}^{2ax} + 2\mathrm{e}^{ax}\sum_{\nu=1}^{\infty}\frac{z_\nu \sin\left(\frac{z_\nu x}{L}\right)}{aL - a^2L^2 - z_\nu^2}\mathrm{e}^{-D\left(\frac{z_\nu^2}{L^2} + \frac{V^2}{4D^2}\right)t}\right). \tag{43}$$

In the case of a horizontal aquifer, $S = 0$, whence $a = 0$ and Eq. (31) simplifies to

$$R_\nu = \frac{-2y\left[\sin(z_\nu x/L) + \frac{lz_\nu}{L}\cos(z_\nu x/L)\right]\mathrm{e}^{s_\nu t}}{z_\nu\left(\frac{l^2 z_\nu^2}{L^2} + \frac{l}{L} + 1\right)}; \tag{44}$$

z_ν are the roots of $\tan z_\nu = L/lz_\nu$, with which the first-order poles s_ν are computed via Eq. (29a).

If the aquifer is horizontal and the streambed sediment layer absent, $a = 0$ and $l = 0$, whence

$$R_\nu = -\frac{2y}{z_\nu}\sin\left(\frac{z_\nu x}{L}\right)\mathrm{e}^{s_\nu t}; \tag{45}$$

in this case $\tan z_\nu \to \infty$ and the roots are periodical

$$z_\nu = \frac{(2\nu - 1)}{2}\pi, \quad \nu = 1,2,3,\ldots,\infty, \tag{46}$$

giving as first-order poles

$$s_\nu = \frac{-Kh_o(2\nu - 1)^2\pi^2}{4nL^2}, \quad \nu = 1,2,3,\ldots,\infty \tag{47}$$

Use of Eqs. (29) and (45)–(47) gives the results for a horizontal aquifer (index h), with $l = 0$, which show a phase shift of $\pi/2$ between the depth and flow profiles:

$$\lim_{S\to 0} h(x,t) = h_h(x,t) = y\left[1 - \sum_{\nu=1}^{\infty}\frac{4}{(2\nu - 1)\pi}\sin\left[\frac{(2\nu-1)\pi}{2}\frac{x}{L}\right]\mathrm{e}^{\frac{-Kh_o(2\nu-1)^2\pi^2}{4nL^2}t}\right], \tag{48}$$

$$\lim_{S\to 0} q(x,t) = q_h(x,t) = 2y\frac{Kh_o}{L}\sum_{\nu=1}^{\infty}\cos\left[\frac{(2\nu-1)\pi}{2}\frac{x}{L}\right]\mathrm{e}^{\frac{-Kh_o(2\nu-1)^2\pi^2}{4nL^2}t}. \tag{49}$$

The corresponding dimensionless expressions are:

$$H_h(X,T) = Y\left\{1 - \sum_{\nu=1}^{\infty}\frac{4}{(2\nu-1)\pi}\sin\left[\frac{(2\nu-1)\pi}{2}X\right]\mathrm{e}^{-\frac{(2\nu-1)^2}{4}\pi^2 D_oT}\right\}, \tag{50}$$

$$Q_h(X,T) = 2YD_o\sum_{\nu=1}^{\infty}\cos\left[\frac{(2\nu-1)\pi}{2}X\right]\mathrm{e}^{-\frac{(2\nu-1)^2}{4}\pi^2 D_oT}, \tag{51}$$

where the non-dimensional variables are as defined in Eq. (6b). Carslaw and Jaeger (1959) derive Eq. (50) by solving the pure conduction equation by operational methods; in addition, Akylas et al. (2006) verify Eq. (50) by the classical methodology of separation of variables. Then, proceeding as before, we get from Eqs. (48) and (50) the SRF in dimensional and in dimensionless form, respectively:

$$u_h(x,t) = \frac{D}{L^2}\sum_{\nu=1}^{\infty}(2\nu-1)\pi\sin\left[\frac{(2\nu-1)}{2}\pi\frac{x}{L}\right]\mathrm{e}^{-\frac{D}{L^2}\frac{(2\nu-1)^2}{4}\pi^2 t}, \tag{52}$$

$$U_h(X,T) = \frac{u(x,t)}{K/nL} = D_o\sum_{\nu=1}^{\infty}(2\nu-1)\pi\sin\left[\frac{(2\nu-1)}{2}\pi X\right]\mathrm{e}^{-D_o\frac{(2\nu-1)^2}{4}\pi^2 T}. \tag{53}$$

Finally, bank storage from flow across the stream bank, Eqs. (40)–(42), specialises to

$$\mathrm{vol}_h(t) = (nyL)\left[1 - \frac{8}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{(2\nu-1)^2}\mathrm{e}^{-\frac{D}{L^2}\frac{(2\nu-1)^2}{4}\pi^2 t}\right] = (nyL)\mathrm{VOL}_h \tag{54}$$

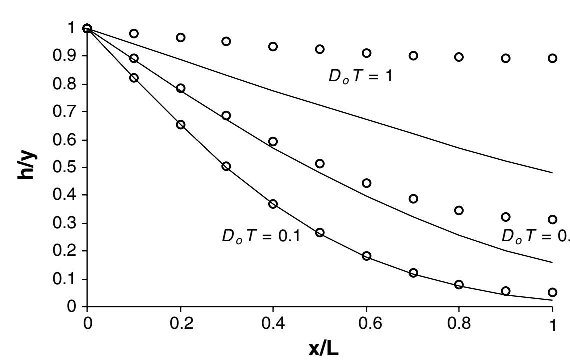

Figure 3 Comparison of analytical response to a step change y [normalised profiles $h(X, T = \mathrm{const.})/y$] in a horizontal aquifer of length $L = 100$ m (○○○○) with that of semi-infinite horizontal aquifer (--) for $D_oT = 0.1$, 0.25 and 1.0; $D_o = H_o = h_o/L = y/2L$.

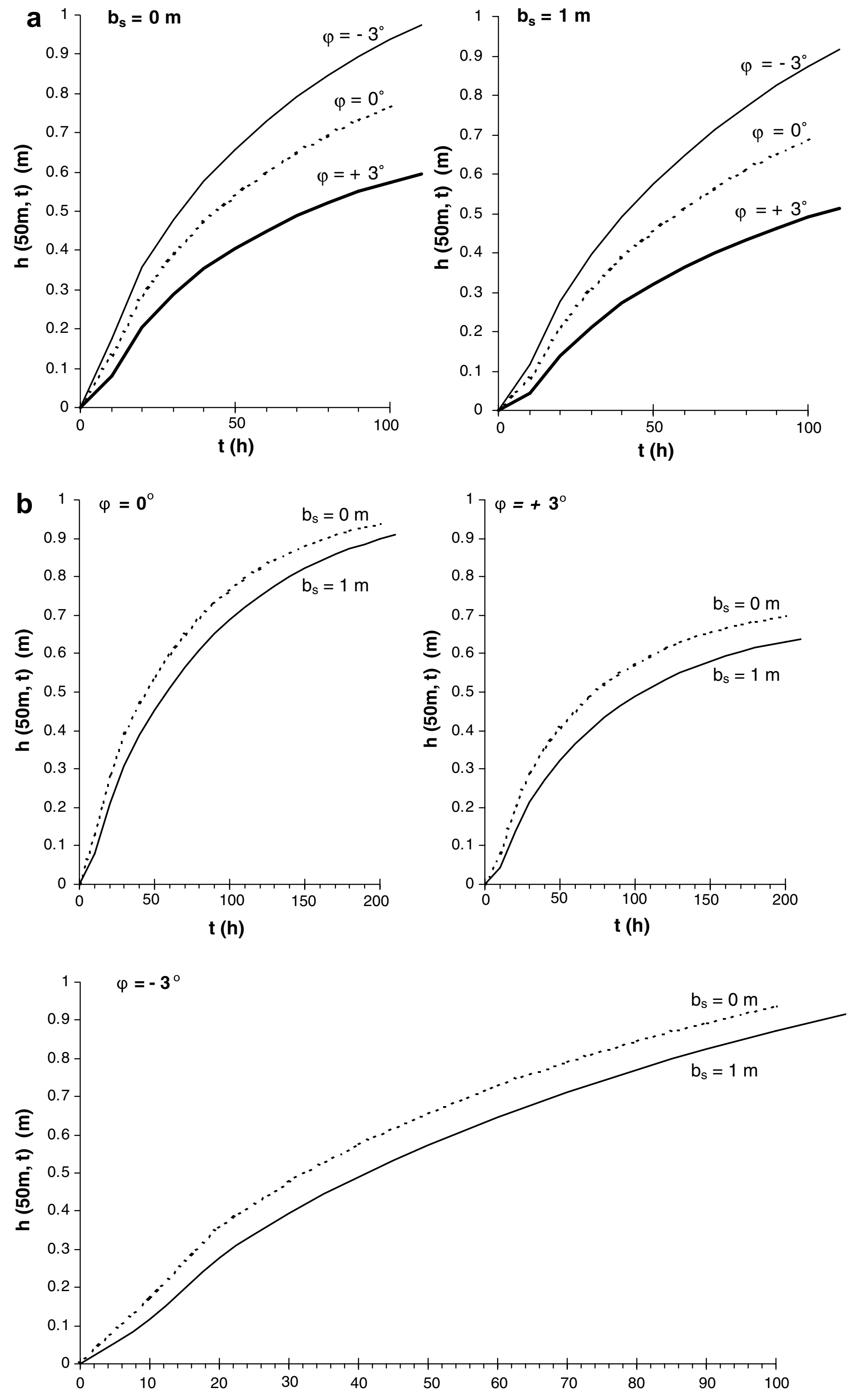

Figure 4 Sensitivity study of flow in aquifers, on base length L = 100 m, inclined at $\varphi = \pm 3°$ and also horizontal, $\varphi = 0°$, as well as with and without sediment streambed layer, b_s = 0 and 1.0 m and $K_s = K/10$ = 2.5 m/day, i.e. l = 0 and 10 m: (a) hydrographs at x = 50 m, for fixed b_s-value, with slope as parameter; (b) hydrographs at x = 50 m, for fixed slope, with streambed thickness as parameter; and (c) depth profiles at t = 100 h, for fixed b_s-value, with slope as parameter.

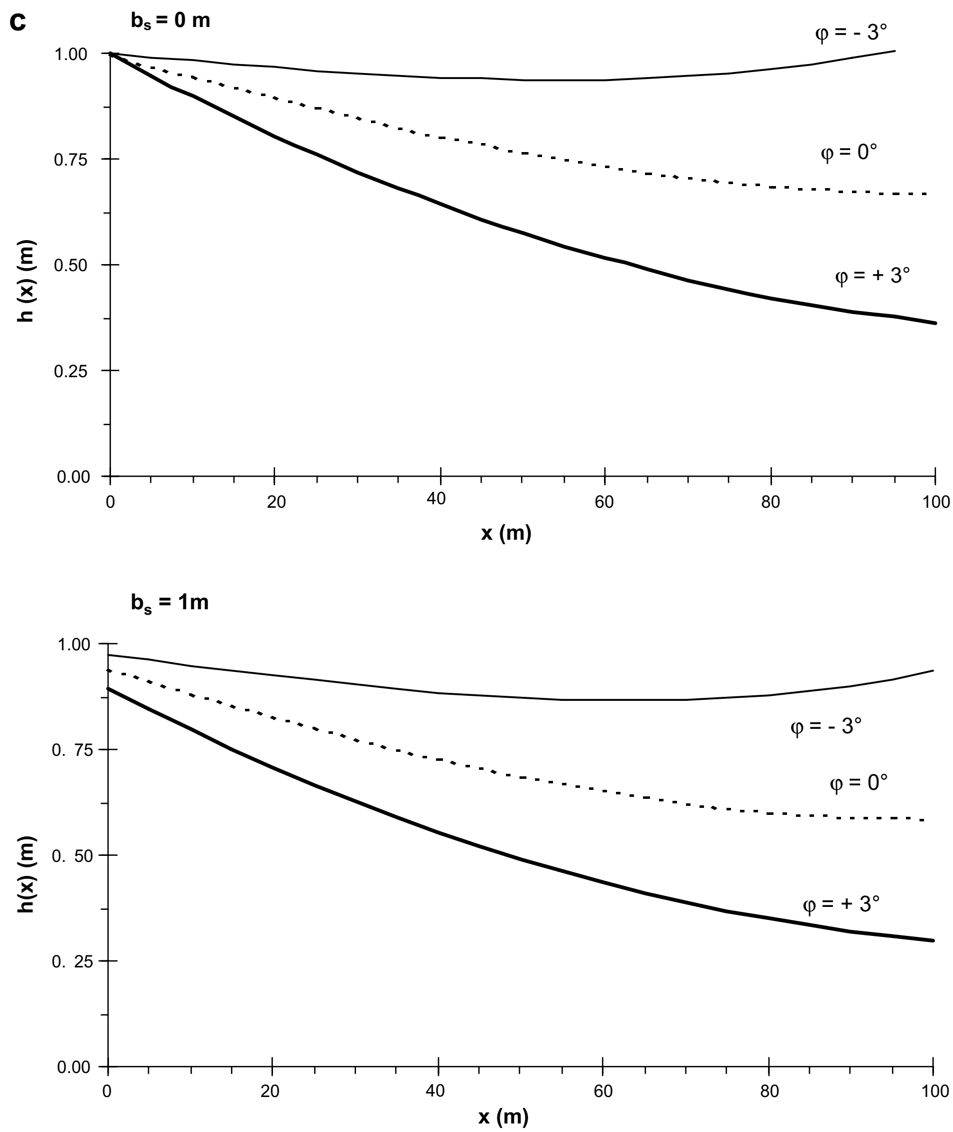

Figure 4 (*continued*)

Note that, at $t = T = 0$, $\exp(s_\nu t) = 1$ and bank storage $\mathrm{vol_h}(0) = \mathrm{VOL_h}(0) = 0$ because the sum of the series is $\pi^2/8$; also, $\mathrm{vol_h}(t \to \infty) = n\,y\,L$, i.e., the aquifer fills to the stream level.

As seen in Fig. 3, Eq. (49) agrees up to $D_oT \approx 0.1$ with the well-known similarity solution for diffusion in a semi-infinite domain, after a step rise Y from a zero initial condition:

$$H_h(X,T) = Y\left[1 - \mathrm{erf}\left(X/\sqrt{4D_oT}\right)\right] = Y\mathrm{erfc}\left(X/\sqrt{4D_oT}\right);$$

$$\mathrm{erf}(\Psi) = (2/\pi^{1/2})\int_0^{\Psi}\exp(-\psi^2)\mathrm{d}\psi. \tag{55}$$

The expressions for the flow rate and the bank storage corresponding to Eqs. (51) and (54) are

$$Q_h(X,T) = Y\sqrt{\frac{D_o}{\pi T}}\exp(-X^2/4D_oT); \quad Q_h(0,T) = Y\sqrt{\frac{D_o}{\pi T}}, \tag{56}$$

$$\mathrm{VOL_h}(T) = Y\sqrt{\frac{4D_oT}{\pi}}. \tag{57}$$

If a stream with a sediment streambed layer ($l \neq 0$) is connected to a horizontal aquifer ($S = 0$), the first derivative term in Eq. (17) vanishes ($V = 0$, $a = 0$), c_1, c_2 change accordingly and the transformed solution Eq. (22) writes

$$F(x) = \frac{y\cosh(bL - bx)}{s[\cosh(bL) - bl\sinh(bL)]}. \tag{58}$$

Moench and Barlow (2000) derived a version of Eq. (58), without inverting it analytically however. Inverting from the s-domain [poles: $s = 0$, s_ν, $\nu = 1,2,\ldots,\infty$, from Eq. (29a) for $V = 0$; z_ν from $\tan z_\nu = L/lz_\nu$], Eqs. (25) and (34) give as time-domain step response and SRF:

$$h_h(x,t) = y\left[1 - 2\sum_{\nu=1}^{\infty}\frac{\sin\left(\frac{z_\nu x}{L}\right) + z_\nu\frac{l}{L}\cos\left(\frac{z_\nu x}{L}\right)}{z_\nu\left(1 + z_\nu^2\frac{l^2}{L^2}\right)}e^{-\frac{Dz_\nu^2}{L^2}t}\right], \tag{59}$$

$$u_h(x,t) = 2\frac{D}{L^2}\sum_{\nu=1}^{\infty}\frac{z_\nu\left[\sin\left(\frac{z_\nu x}{L}\right) + z_\nu\frac{l}{L}\cos\left(\frac{z_\nu x}{L}\right)\right]}{\left(1 + z_\nu^2\frac{l^2}{L^2}\right)}e^{-\frac{Dz_\nu^2}{L^2}t}. \tag{60}$$

Sensitivity study

The following example demonstrates primarily the influence of the slope on the aquifer response. The initial stream depth, $h_{is} = 10$ m, changes abruptly by $y = 1$ m. This 1-m change appears small, if the stream is assumed to, indeed, fully penetrate the aquifer. If, however, the stream penetrates the aquifer by, say, 25% (as in the field case modelled in our companion paper, Koussis et al., in review), the relative stream depth change for the 1-m stage variation is 40% (1 m/2.5 m). The aquifer parameters are: $L = 100$ m, $K = 25$ m/day, $n = 0.2$, angle of base inclination $\varphi = 0°$ and $\pm 3°$; the conductivity of the streambed is $K_s = 2.5$ m/day. To study the sensitivity to the leakance, we consider sediment bed thickness values $b_s = 0$ and 1.0 m, with respective leakances $l = 0$ and 10 m. We would like to also note that, as a result of geologic processes (faults, lifting and erosion), negative base slopes, indeed, occur in nature.

The linearisation depth can be estimated by Eq. (16) (or, more roughly, as the mean depth of the horizontal water surface profile). But these steady-state estimates should be adjusted for transient flow. In the case of a step input, we may simply take the estimate from the steady state solution for $h_s(0) = h_{is} + y/2$. Generally however, appropriate values should take into account the character of a transient, as discussed in the companion paper Koussis et al. (in review). The linearisation depth depends also on the leakance, which approximately extends the aquifer length to an equivalent $L + l$, to account for the added flow resistance (steady state) by the streambed layer. In this example the values differ slightly relative to the case without a streambed layer: $\varphi = 3°$, $h_o \cos\varphi = 8.62$ m for $l = 0$ versus 8.52 m for $l = 10$ m; $\varphi = -3°$, $h_o \cos\varphi = 12.11$ m for $l = 0$ versus 12.28 m for $l = 10$ m.

Fig. 4a shows water level hydrographs (relative to the initial steady state) at the mid-point of the aquifer, $x = 50$ m, with $\varphi = \pm 3°$ as curve parameter for $b_s = 0$ and 1.0 m ($l = 0$ and 10 m) and Fig. 4b shows the hydrographs with $b_s = 0$ and 1.0 m as curve parameter for $\varphi = \pm 3°$; the hydrographs for the horizontal aquifer, linearised properly at $h_o = 10.5$ m, are also included for reference. Finally, Fig. 4c shows aquifer level profiles (relative to the initial condition), at $t = 100$ hours. In all figures, water levels in excess of the stream level can occur; as already discussed, these are owed to the chosen optimisation criterion for the specification of the linearisation depth.

The differences among the hydrographs due to the slope are significant, indicating that the inclination of the aquifer base should not be ignored, even at relatively small angles. We observe that the water level discrepancies are stronger between an aquifer inclined at $\varphi = +3°$ and the horizontal aquifer than between an aquifer inclined at $\varphi = -3°$ and the horizontal aquifer. The leakance, $l = b_s K/K_s$, is seen in Fig. 4 to play also an important role in the response of an aquifer, influencing the behaviour of the aquifer most on positive slopes. In the case of $\varphi = -3°$, we observe that the aquifer levels for $l = 0$ initially exceed those obtained when leakance is included; this order is reversed in later times and the curves cross. This behaviour is explained as follows: at early times, in the absence of a streambed layer (no added resistance), the stream water enters the aquifer more readily than when a sediment layer is present. At later times, however, the solution is dictated increasingly by the steady flow conditions, which, in the case of $\varphi < 0$, give higher water levels when a streambed layer exists.

Summary

We derive Laplace transform solutions, which we invert analytically, for the interaction of a stream with a sloping unconfined aquifer that it penetrates fully. These solutions are based on the linearised 1-D Boussinesq equation (Dupuit assumption), extended for a sloping aquifer base; an impervious barrier bounds the aquifer at its landward end. The validity of the linear hydraulic model is critiqued and an objective definition of the linearisation level is developed. The solutions account for a low-conductivity streambed layer, when such a layer is present, and treat the horizontal aquifer case in the zero-slope limit. Unit-impulse responses are also derived that can be used to solve complex problems through convolution. In a special case, we verify our solution against known analytical results for step inputs. Significant differences of responses to step inputs are found for aquifers on $\varphi = -3°$, $\varphi = 0°$ and $\varphi = +3°$ base slopes, and for a stream with and without a low-conductivity bed layer.

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