Stationarity of linearly forced turbulence in finite domains

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A simple scheme of forcing turbulence away from decay was introduced by Lundgren some time ago, the "linear forcing," which amounts to a force term that is linear in the velocity field with a constant coefficient. The evolution of linearly forced turbulence toward a stationary final state, as indicated by direct numerical simulations (DNS), is examined from a theoretical point of view based on symmetry arguments. In order to follow closely the DNS, the flow is assumed to live in a cubic domain with periodic boundary conditions. The simplicity of the linear forcing scheme allows one to rewrite the problem as one of decaying turbulence with a decreasing viscosity. Its late-time behavior can then be studied by scaling symmetry considerations. The evolution of the system in the description of "decaying" turbulence can be understood as the gradual symmetry breaking of a larger approximate symmetry to a smaller symmetry that is exact at late times. The latter symmetry implies a stationary state: In the original description all correlators are constant in time, while, in the "decaying" turbulence description, that state possesses constant Reynolds number and integral length scale. The finiteness of the domain is intimately related to the evolution of the system to a stationary state at late times: In linear forcing there is no other large scale than the domain size, therefore, it is the only scale available to set the magnitude of the necessarily constant integral length scale in the stationary state. A high degree of local isotropy is implied by the late-time exact symmetry, the symmetries of the domain itself, and the solenoidal nature of the velocity field. The fluctuations observed in the DNS for all quantities in the stationary state can be associated with deviations from isotropy that is necessarily broken at the large scale by the finiteness of the domain. Indeed, to strengthen this conclusion somewhat, self-preserving isotropic turbulence models are used to study evolution from a direct dynamical point of view. Simultaneously, the naturalness of the Taylor microscale as a self-similarity scale in this system is emphasized. In this context the stationary state emerges as a stable fixed point. We also note that self-preservation seems to be the reason behind a noted similarity of the third-order structure function between the linearly forced and freely decaying turbulence, where, again, the finiteness of the domain plays a significant role.

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I. INTRODUCTION

Maintaining a turbulent flow in a more or less stationary state, for better statistics in experiment or convenience in theoretical considerations, requires forcing the flow, that is, feeding it energy that balances dissipation happening at the smallest scales. In numerical simulations of incompressible isotropic turbulent flows one usually solves the Navier-Stokes equations in a cubic box (with periodic boundary conditions). For an account of direct numerical simulation (DNS) methods, see Ref. [1]; for a recent review on the current isotropic turbulence statistics from DNS, see Ref. [2]. In most cases, forcing takes the form of a force term in wave number space (spectral space) that vanishes for all but the smaller wave numbers, i.e., one feeds energy at the largest scales of the turbulent flow in the box. The general concept is that the details of the larger scales are model dependent but the details of all other scales, that is, those where some universal laws may hold, depend only on the intrinsic dynamics of the Navier-Stokes equations, at least for high Reynolds numbers. Presumably, by forcing turbulence, one achieves satisfactory results for given a resolution for higher Reynolds numbers than in the freely decaying turbulence.

There have been developed various kinds of forcing schemes. The simpler ones fiddle in a suitable manner the magnitude of velocity field, or the total energy of the lower wave number modes, imitating an energy input in the larger scales [3–6]. These models can be regarded as essentially deterministic in the sense that that there is no additional randomness introduced in the problem. There are also deterministic models that explicitly introduce a force term in the Navier-Stokes equations, whose details are either postulated or derived by a postulated auxiliary model [7–10]. In stochastic forcing models [11–13] the details of the force term are determined by additional random variables following prescribed stochastic processes. Each of those models suffers from one set or more sets of problems, such as excessive fluctuations around stationarity, relatively long relaxation period to stationarity, persistent anisotropy, excessive distortion of large-scale motions, and introduction of irrelevant features in the description of turbulence. A useful comparative discussion between certain deterministic and stochastic models can be found in Ref. [14].

Lundgren proposed in Ref. [15] that we may simplify the deterministic models to the bare minimum, in some sense, assuming that the usually velocity dependent force term is merely proportional to the velocity field for all positions $\mathbf{x}$, or all wave numbers $\mathbf{k}$, and all times: $\mathbf{f} = A\mathbf{u}$, where A is plainly a constant. The "linear forcing" scheme was further studied in Refs. [16,17]. Its simple force term $A\mathbf{u}$ has the same form

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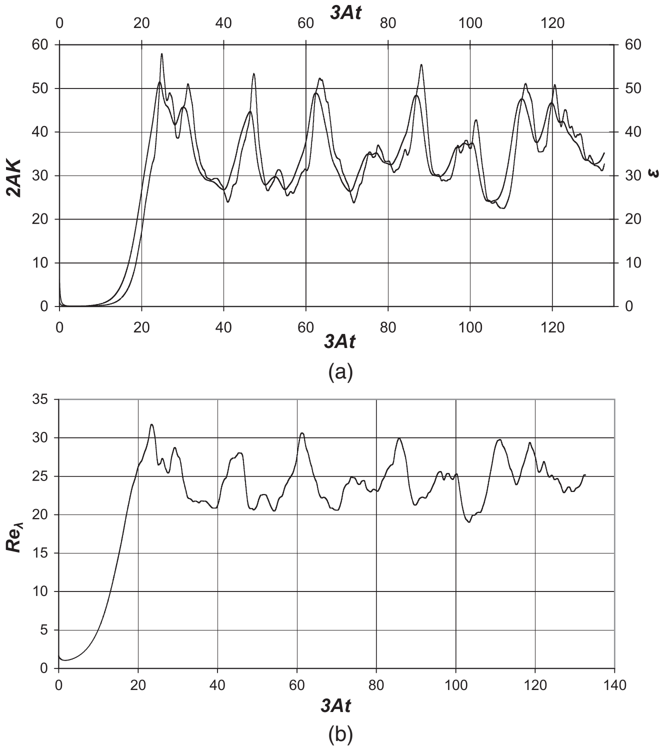

FIG. 1. A typical evolution of the energy production rate (solid line) and dissipation rate (dashed line) is shown in (a) and Taylor microscale Reynolds number in (b). The parameters chosen are $A = 1$, box size $l = 2\pi$, and viscosity $\nu = 0.1$.

in both the spectral and physical spaces. Thus, unlike other forcing schemes, it may be used equally well in cases that need to be solved directly in the physical space with boundary conditions that differ from periodic ones [16]. That feature could prove useful. Additionally, although in linear forcing the injection of energy into the flow is not restricted to the larger scales, this scheme performs decently and, in fact, possibly better in the region between the inertial range and the integral scale than the other forcing schemes in Ref. [15]. From the theoretical point of view, what matters most is that, unlike limited spectral bandwidth forcing schemes, linear forcing does not introduce an additional length scale in the problem at the level of the Navier-Stokes equations (a length scale outside the equations is, of course, introduced by the boundary conditions).

The performance of the linear forcing scheme with respect to its convergence properties was studied in considerable detail in Ref. [16] and useful remarks were made in Ref. [14]. The clear conclusion is that linear forcing results in relatively large fluctuations in the stationary phase. Indeed, a typical evolution of the energy production rate $2AK$ (where K is the total kinetic energy per unit mass), the dissipation rate ε, and the Taylor microscale Reynolds number Re_λ is shown in Fig. 1. (The details of the DNS can be found in Ref. [17].) From the practical point of view this is a disadvantage as it requires longer simulations in order to obtain good statistics. Moreover, the stationary state is reached after a relatively long transient period [14,16], requiring even more computational time. On the other hand, linear forcing leads to quite controllable situations in the stationary state: Given the scales of the problem, i.e., the rate A, the cubic box size l, and the viscosity ν, the facts of the stationary state are predictable. The balance between the energy production and dissipation, $2AK = \varepsilon$, is indeed observed on the (time-)average, validating the very concept of a stationary state; the dissipation length $L_\varepsilon = (2K)^{\frac{3}{2}}/\varepsilon$ turns out to be equal to the box size l within few percentage points of error in all cases [16]. The Reynolds number $\mathrm{Re}_L = K^2/(\varepsilon\nu)$ may be rewritten as $\frac{1}{4}AL_\varepsilon^2/\nu$ at the stationary state and should then be roughly equal to $\frac{1}{4}$ of the natural order of Re_L in this problem, Al^2/ν, in all cases, as isobserved [17]. For example, the Taylor microscale Reynolds number $\mathrm{Re}_\lambda = (\frac{20}{3}\mathrm{Re}_L)^{\frac{1}{2}}$ is expected to be roughly equal to 25.7 for the run shown in Fig. 1. Indeed, the average of the Re_λ time series in Fig. 1 differs by only few percentage points from that estimate.

Even if we take stationarity for granted, its characteristics, i.e., the relatively large fluctuations and the "predictability" of quantities describing the state of turbulence, certainly call for understanding. On the other hand, the very existence of a stationary state in this scheme is a fairly intriguing matter. The long-time effect of the energy production competing with dissipation is not, *a priori*, clear. From the dynamical point of view, it is clear that the dissipation term $\nu\nabla^2 u$ becomes stronger than the force term Au at scales smaller than $(\nu/A)^{\frac{1}{2}} \sim \mathrm{Re}_\lambda^{-1} l$, but it is not clear whether energy that is produced at all other scales up to l will be dissipated by an adequate rate at those smaller scales.

We will approach the problem as follows. The relative simplicity of linear forcing allows us to study its late-time evolution employing scaling symmetry arguments to an extent enjoyed possibly only in freely decaying turbulence; in fact, as we shall show, there is a relationship between linearly forced and freely decaying turbulence. A parallel discussion between them can be made. Sections II–V will be devoted to presenting these arguments. The predictability, as we called it above, of the stationary state, is enlightened through those symmetry arguments, essentially on the basis that there is no intrinsic large length scale in the dynamical equations apart from that introduced by the boundary conditions, i.e., the finite size l of the domain. The remaining question, then, is why the fluctuations observed in the stationary state, as seen, e.g., in the Fig. 1, are so large. We shall argue, as analytically as we can, that the fluctuations can be associated with the deviations from isotropy accumulated by this forcing at all scales between the scale $(\nu/A)^{\frac{1}{2}}$ and the domain size l (unlike the limited bandwidth forcing schemes that feed anisotropy only at the domain size scale where isotropy is already broken). The method we shall use is to reduce the dynamical problem to a two-equation model. As a cross-check of our previous conclusions, the stationary state re-emerges as a stable fixed point of the evolution, a by-product of which is that fluctuations tend to be suppressed as long as turbulence is isotropic. This part of our discussion is presented mostly in Secs. VI and VII. It is interesting to note that, from various aspects, linearly forced turbulence seems to be a natural context for the direct application of various ideas that have been developed in the study of freely decaying turbulence, in fact, one may dare say, an even more natural context.

In terms of equations, a linearly forced incompressible flow with *zero mean flow* velocity is described by the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + A\mathbf{u}\,, \tag{1}$$

where the incompressibility condition reads $\nabla\cdot\mathbf{u}=0$; the velocity field is solenoidal. A is a positive constant with dimensions of inverse time. The term $A\mathbf{u}$ is a curious "antidrag" force on fluid particles. (If there is a mean flow with velocity $\mathbf{U}$, the force term should read $A(\mathbf{u}-\mathbf{U})$, i.e., it involves the fluid velocity relatively to the mean flow. Presumably, the forcing does not break Galilean symmetry. If that were not the case, the forcing would lead to physical pathologies; see, e.g., Ref. [18].) As already mentioned, we impose periodic boundary conditions: $\mathbf{u}(x,y,z)=\mathbf{u}(x,y+l,z)=\mathbf{u}(x,y,z+l)=\mathbf{u}(x+l,y,z)$. That is, the flow evolves within a cubic domain with a side equal to l, obeying the given conditions on its boundary. (We will often refer to the cubic domain simply as the *box*.) As we shall emphasize later, the term *cubic domain* is slightly misleading due to the periodic boundary conditions: The flow essentially evolves in a boundary-less space of finite size. There are no walls anywhere, and this is why we describe the domain as finite instead of bounded. The problem we are interested in to determine the late-time state of the turbulent flow governed by these equations and conditions.

The present work is organized as follows. In Secs. II and III a reformulation of the problem and an associated scaling symmetry are presented. In Sec. IV the implications of the scaling symmetry and of the symmetries of the domain for the late-time behavior of the ensemble average correlators are discussed. In Sec. V we restrict ourselves to isotropic turbulence to argue in a more detailed manner for the stationary state as the final phase of the linearly forced turbulence, as described by the exact ensemble average correlators and taking into account the effects of the finiteness of the domain. In Sec. VI the expected behavior of the actual observables in DNS, i.e., the box-averaged correlators, is discussed in relation to the properties of the ensemble average correlators established in the previous sections. In Sec. VII we combine the powerful condition of isotropy with the (by now established) existence of fluctuations around stationarity: A complete self-preserving isotropic turbulence model is obtained and applied to study the fate of fluctuations at scales in the flow where isotropy holds. We close by summarizing our work and by touching on some other interesting aspects of linearly forced turbulence, discussing also certain open issues of the problem in the final section.

II. UNFORCED TURBULENCE WITH DECAYING VISCOSITY

We shall proceed as follows. Mathematically, we may rewrite the problem as an equation for a new field $\mathbf{u}'$:

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \nu'\nabla^2\mathbf{u}', \tag{2}$$

with $\mathbf{u}'$ being related to $\mathbf{u}$ by

$$\mathbf{u}' = F\,\mathbf{u}, \tag{3}$$

where F is a function of time $F=F(t')$ to be determined and $\nabla\cdot\mathbf{u}=0$ implies that $\nabla\cdot\mathbf{u}'=0$.

Substituting (3) into (2) we get

$$\frac{1}{F^2}\frac{dF}{dt'}\mathbf{u} + \frac{1}{F}\frac{\partial\mathbf{u}}{\partial t'} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{F^2}\frac{1}{\rho}\nabla p' + \frac{1}{F}\nu'\nabla^2\mathbf{u}. \tag{4}$$

This equation becomes identical to Eq. (1) on setting

$$F dt' = dt, \quad \frac{1}{F^2}\frac{dF}{dt'} = -A, \quad \nu' = F\nu. \tag{5}$$

The transformation of pressure, $p'=F^2p$, follows from its Laplace equation constraint and cannot be regarded as an independent condition.

For constant A we have from the first two differential equations above that

$$F = e^{-A(t+t_o)} \quad \text{and} \quad A(t'+t_o') = e^{A(t+t_o)}, \tag{6}$$

where t_o and t_o' are integration constants.

Viscosity ν' reads

$$\nu' = \frac{1}{A(t'+t_o')}\nu, \tag{7}$$

in the unphysical time coordinate t'. The problem has become unforced turbulence with decaying viscosity. The way it decays, $\propto 1/t'$, is crucial in what follows.

Note that everything can be transformed back to the initial variables at all times, except $t=\infty$ or $t'=\infty$. This is a singular point of the transformation as F vanishes and the two forms of the problem are not equivalent. This would be a relevant subtlety only if we had to deal with the actual limit $t\to\infty$. We will not need such a limit anywhere in our analysis.

It also worth noting that a transformation $t\to t'$ can generate only a term that is linear in $\mathbf{u}$, unless the transformation depends itself on the velocity field. Therefore, the *possibility* of such a generating transformation is intimately related to linear forcing.

III. TIME-TRANSLATION INVARIANCE AND AN EXACT SCALING SYMMETRY

Note that Eq. (1) does not explicitly depend on time, forcing being a function of the velocity field alone. Therefore, depending on boundary conditions, this equation may permit nontrivial solutions that are static or independent of time in a certain sense. As we are interested in fully developed turbulence, the time dependence in question will apply to statistically defined quantities.

The Navier-Stokes equations we arrived at by transforming to time t' in the previous section explicitly reads

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\nabla)\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \frac{\nu}{A(t'+t_o')}\nabla^2\mathbf{u}', \tag{8}$$

where ρ, ν, and A are constants and $\nabla\cdot\mathbf{u}'=0$. Apart from the fairly peculiar time dependence of viscosity, that is, the energy dissipation rate decreases with time, the form of this equation is more familiar than that of Eq. (1).

The time-translation invariance of Eq. (1) now translates to an exact scaling symmetry of Eq. (8). Even by inspection one may verify that the transformation

$$t' + t'_o \to e^a(t' + t'_o), \qquad \mathbf{u}' \to e^{-a}\mathbf{u}', \tag{9}$$

for any constant a is an exact symmetry of the previous equation (necessarily, $p' \to e^{-2a}p'$).

Formulas will take a more attractive form, more in line with our purposes, as we shall explain below, if we drop the arbitrary integration constant t'_o, for example, by redefining the time coordinate t', and write the symmetry in the form

$$t' \to e^a t', \qquad \mathbf{u}' \to e^{-a}\mathbf{u}'. \tag{10}$$

The constant t'_o can be set to a specific value only by additional conditions, most probably associated with the early stages of the evolution of the flow. This will be the case only in Sec. V C. In fact, as our reasoning will refer mostly to the fully developed stage of turbulence, that is, at late times that lie far away from the initial conditions, any such constant t'_o can be equivalently kept or dropped, or even changed to another convenient value, at an increasingly good approximation as the flow evolves.

The origin of the late-time scaling symmetry is clear: Shifting t means rescaling t'. Shifting t is a symmetry of Eq. (1), therefore rescaling t' must be a symmetry of Eq. (8), as it is indeed the case. It is important to remember that the symmetry (10) respects the periodic boundary conditions on the field $\mathbf{u}'$, therefore, it is an exact symmetry of the problem.

We may now forget Eq. (1) for a little while and focus on the unforced turbulence described by Eq. (8). Its scaling symmetry will allow us to draw certain conclusions about the late-time behavior of the system.

IV. SCALING SYMMETRIES, ASYMPTOTIC BEHAVIOR, AND ISOTROPY

In order to get a first idea why the symmetry can be useful that way, note that the product $t'\mathbf{u}'$ is invariant under the scaling (10). Consider an arbitrarily chosen moment of time t'_0 and the velocity field $\mathbf{u}'_0$ at that moment and another moment $t' = e^a t'_0$ when velocity is $\mathbf{u}'$. Invariance means $t'\mathbf{u}' = t'_0\mathbf{u}'_0$. Equivalently, we may write

$$\mathbf{u}' = \frac{1}{t'}\, t'_0\, \mathbf{u}'_0. \tag{11}$$

In general, a symmetry transformation moves us around the space of solutions. That is, all the previous relation means is that if there is a solution with velocity $\mathbf{u}'_0$ at time t'_0, then there is another solution with velocity field $\mathbf{u}'$ at time t', i.e., in general, $\mathbf{u}'$ and $\mathbf{u}'_0$ need not necessarily correspond to the same initial conditions.

On the other hand, the symmetry holds for *large* times t' and t'_0. Even if it did not, that would be a convenient choice for the following reason. The initial time $t' = 0$ is pushed into the remote past, and the behavior (11) might then be an exact *asymptotic* result for a large class of solutions, meaning irrespectively of their initial conditions. That implies that $t'\mathbf{u}' = t'_0\mathbf{u}'_0$ is an actual constant at each point $\mathbf{r}$ in space depending only on the parameters of the equation and the boundary conditions.

The constant in question is a vector. To be more specific, recalling that $\mathbf{u}'$ satisfies $\nabla \cdot \mathbf{u}' = 0$, we need a solenoidal vector field in steady state that does not depend on initial conditions; i.e., it is unique. Such a field must respect the symmetries of the boundary conditions, that is, the symmetries of the cube. There is no such thing: solenoidal vector fields have closed integral curves that can always be reversed by reflections. We deduce then that (11), as long as it is nontrivial, will always depend to some extend on t'_0, i.e., on initial conditions. Thus it is not of much use in this form.

Our reasoning can be used more effectively if it is applied in statistically defined quantities, that is, correlators of the velocity field. As mentioned in the Introduction, between this section and Sec. VI we shall work with correlators defined as averages over a statistical ensemble. The statistical ensemble averages are independent of the initial conditions by their very definition: They are averages over the space of solutions. Of course, in a problem on turbulence they certainly are the quantities of interest. The statistical ensemble averages will be denoted with an overbar.

Equation (11) then holds trivially for no mean flow: $\overline{\mathbf{u}'} = 0$. One then considers general correlators of the velocity field, $\overline{u'_{i_1}(\mathbf{r}_1,t'_1)u'_{i_2}(\mathbf{r}_2,t'_2)\cdots}$, and their derivatives. Consider a correlator $T'_{j_1 j_2 \ldots}(\vec{r},t')$ that involves the velocity field or its derivatives n times. Let such a tensor field with n velocity field insertions in the correlation. Symmetry (10) then tells us, similarly to Eq. (11), that

$$T'_{j_1 j_2 \ldots} = \frac{1}{t'^n}\, t_0'^n\, T'_{0\, j_1 j_2 \ldots}. \tag{12}$$

Now $t_0'^n\, T'_{0\, j_1 j_2 \ldots}(\mathbf{r},t'_0)$ must be a constant at each point $\mathbf{r}$ in space. If not, then this quantity does depend on t'_0, i.e., on the initial conditions. This means that this quantity is not well defined as a ensemble average, i.e., it mathematically does not exist and it must be defined in an approximate manner that does not possess the expected properties or only approximates them. The reason this may happen is that the system has not reached a stage where ensemble averages are meaningful; *a priori*, some kind of equilibrium is required.

Now, same as with $t'_0\mathbf{u}'_0$, most of these constant tensor fields must be zero by being inconsistent with the symmetries of the cube (especially reflections) and the incompressibility condition. Certainly everything with at least one solenoidal index must vanish. This leaves us with the scalars, the tensors manufactured from them and the Kronecker δ, and correlators such as $\overline{\partial_i u_k \partial_j u_k}$ with no free solenoidal index.

In order to see an example of how these statements are realized, consider the correlator $t_0'^2\, \overline{u'_{0i}u'_{0k}}$, which is constant in time. Being constant in time means that it must respect the symmetries of the cubic domain: It must not change under reflections of the domain around planes of symmetry and rotations around axis of symmetry. One should recall that our correlators are ensemble averages over the whole of phase space, thus symmetries cannot take us to an other constant late-time solution: There is no other solution or we have convergence problems in the very definition of our averages. It is then easy to see that $t_0'^2\, \overline{u'_{0i}u'_{0k}}$ must be equal to $\delta_{ik}\, t_0'^2\, \overline{u_{0i}' u_{0i}'}(\text{no sum}) = \frac{1}{3}\delta_{ik}\, t_0'^2\, \overline{u_{0j}' u_{0j}'}$, i.e., essentially a scalar. Moreover, by the incompressibility condition $\nabla \cdot \mathbf{u}' = 0$ we see that the scalar itself must be constant in space.

One should note that the situation very much resembles that of isotropic, i.e., also homogeneous, turbulence. There is anisotropy allowed by the problem but it is much less than what we would, in general, call anisotropy. Thus, we will proceed by assuming isotropy and analyze what that implies; then, as isotropy cannot hold at scales comparable to the cubic box size l, the effects of the boundary eventually play a key role. This is done in the next section. We close this section by defining a few important scalars for the description of turbulence, their symmetry and transformation properties, and their expected late-time behavior according to our arguments.

The rms value q of the velocity and the dissipation rate ε are defined by $q^2 = \overline{\mathbf{u} \cdot \mathbf{u}}$ and $\varepsilon = \nu\, \overline{\partial_j u_i \partial_j u_i}$. Moreover, by $K = \frac{1}{2}q^2$ we shall denote the total kinetic energy per unit mass. Similar expressions hold for the primed quantities.

Under the symmetry (10) the quantities K' and ε' transform as

$$K' \rightarrow e^{-2a} K' \qquad \text{and} \qquad \varepsilon' \rightarrow e^{-3a} \varepsilon', \tag{13}$$

where one should bear in mind that ε' involves ν' defined in Eq. (7). Following again the reasoning given in the previous paragraphs we conclude that for large times t' the kinetic energy and dissipation rate should obey

$$K' = \frac{\text{const}}{t'^2} \qquad \text{and} \qquad \varepsilon' = \frac{\text{const}}{t'^3}. \tag{14}$$

In order to see what this result means back in the variables of the system (1), we use Eqs. (3) and (7) to obtain the transformations of K and ε:

$$K' = (At' + 1)^{-2} K \qquad \text{and} \qquad \varepsilon' = (At' + 1)^{-3} \varepsilon. \tag{15}$$

The result is then that the kinetic energy and dissipation rate in the linearly forced turbulence should at late times become

$$K = \text{const} \qquad \text{and} \qquad \varepsilon = \text{const}. \tag{16}$$

Presumably, the dissipation length scale L_ε and the Reynolds number Re_L defined by

$$L_\varepsilon = \frac{q^3}{\varepsilon} \qquad \text{and} \qquad \mathrm{Re}_L = \frac{K^2}{\varepsilon \nu}, \tag{17}$$

and transforming by use of

$$\mathrm{Re}_L' = \mathrm{Re}_L \qquad \text{and} \qquad L_\varepsilon' = L_\varepsilon, \tag{18}$$

should also reach constant values. That is, turbulence should reach what we have already called the stationary state or phase.

The arguments given above can be rephrased in the actual time t and the variables of Eq. (1) as follows. We have already mentioned that shifting time t is a symmetry of Eq. (1). That is, if $\mathbf{u}(t)$ is a solution of this equation then so is $\mathbf{u}(t + \Delta t)$ for an arbitrary interval Δt. These two solutions do not coincide because they correspond to the different initial conditions. On the other hand, we may say that for a certain class of initial conditions that difference should become irrelevant at late times, i.e., the two solutions, or at least certain quantities calculated out of them, will coincide. But this is the same as stating the obvious fact that static or stationary solutions of Eq. (1) exist, without explaining whether such stationary states are indeed the end point of solutions for an reasonably large class of initial conditions. In this light, our arguments as given so far seem rather trivial.

Our arguments are essentially about symmetries. The most convenient context to discuss them, and possibly the only context, is that of isotropic turbulence. We shall argue that the evolution of the system to the stationary phase can be thought of as the gradual breaking of a larger approximate symmetry to the smaller exact symmetry (10), which is solely consistent with the stationary state. That will be realized in certain convenient cases where one may convince oneself that one "watches" the system evolving as claimed. By the very form of Eq. (8) one may guess that standard knowledge from the freely decaying turbulent flows could prove useful to us.

We start by reviewing certain useful facts about the freely decaying isotropic turbulence.

V. HOMOGENEOUS AND ISOTROPIC TURBULENCE

A. Important quantities and formulas

Consider homogeneous and isotropic turbulence. The one-direction rms value of the velocity, q_1, does not depend on the direction, i.e., $q^2 = 3q_1^2$. The two-point correlation function of the velocity is reduced to a scalar $f(r)$ that depends only the distance r between the two points: $\overline{u_l(0) u_l(\mathbf{r})} = q_1^2 f(r)$. All the information of the two-point correlation is contained in components u_l longitudinal in the direction of separation. Moreover, the two-point triple correlation of the velocity can have only longitudinal components and is expressed in terms of a scalar $h(r)$ by $\overline{u_l(0) u_l(0) u_l(\mathbf{r})} = q_1^3 h(r)$. *A priori*, all quantities depend on time, and, for that reason, time dependence is left understood.

Equation (1) with $A = 0$ is the unforced Navier-Stokes equation describing turbulence in the freely decaying state. The "Karman-Howarth equation" [19,20] derived from it under the conditions of homogeneity and isotropy reads

$$\frac{\partial}{\partial t}(q_1^2 f) = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(q_1^3\, h + 2\nu\, q_1^2\, \frac{\partial f}{\partial r}\right)\right]. \tag{19}$$

In freely decaying turbulence the rate at which energy K is decreasing equals the dissipation rate ε, expressing the balance of total energy in that problem.

$$\dot{K} = -\varepsilon. \tag{20}$$

Presumably, this also holds if the viscosity ν depends explicitly on time. This fact will be useful below.

The integral scale, $L \equiv \int_0^\infty f\, dr$, is of the order of magnitude of the dissipation length L_ε. The Taylor microscale λ_g is defined by a differential relation involving f:

$$\lambda_g^2\, \frac{\partial^2 f}{\partial r^2}\bigg|_{r=0} = -1, \quad \text{i.e.,} \quad \lambda_g = \sqrt{\frac{10\nu K}{\varepsilon}}. \tag{21}$$

For completeness, and as we shall briefly need it later, we write down the energy balance equation for the spectral densities of K and ε. It is a Fourier transform of the Karman-Howarth equation (19), see, e.g., Ref. [20]:

$$\partial_t E(k) = -\partial_k T(k) - 2\nu k^2 E(k). \tag{22}$$

The "spectrum" $E(k)$ suitably integrates to give the kinetic energy and dissipation rate, $K = \int_0^\infty E(k)dk$ and $\varepsilon = 2\nu \int_0^\infty k^2 E(k)dk$. $T(k)$ is the spectral energy flux and vanishes for vanishing and infinite wave numbers. Clearly (20) follows by integrating (22) over all k, although the derivation of energy balance equations will be discussed in more detail in Sec. VI.

B. Scaling symmetries and power laws

The scaling arguments given in this section are borrowed from Ref. [21]. The method is an application of the reasoning presented in Sec. IV.

Consider the Karman-Howarth equation (19). Now perform the two-parameter scaling transformation

$$\begin{aligned} t &\to e^a t, \\ r &\to e^b r, \\ q &\to e^{b-a} q, \\ f &\to f, \\ h &\to h, \end{aligned} \tag{23}$$

for arbitrary constants a and b. Changing a for fixed t amounts to time evolution from the initial moment t. Similarly, changing b for fixed r amounts to looking at larger distances. Under (23) Eq. (19) becomes

$$\frac{\partial}{\partial t}(q_1^2 f) = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(q_1^3 h + e^{a-2b}\, 2\nu\, q_1^2 \frac{\partial f}{\partial r}\right)\right]. \tag{24}$$

Consider high Reynolds numbers. The viscosity term then can be dropped. We see that the transformation (23) is an *approximate* symmetry of Eq. (19) for high Reynolds numbers; it can be regarded as a symmetry of the system for infinite Reynolds numbers. Consider, then, quantities of interest such as the kinetic energy K or the integral scale L (equivalently, the dissipation length L_ε). They transforms same as q^2 and r, respectively.

The one-parameter subgroup of the transformation (23) such that

$$\gamma = \frac{b}{a} \tag{25}$$

is an arbitrary but fixed number, given explicitly by

$$\begin{aligned} t &\to e^a t, \\ r &\to e^{\gamma a} r, \\ q &\to e^{\gamma a - a} q, \\ f &\to f, \\ h &\to h, \end{aligned} \tag{26}$$

for arbitrary a, leaves the quantities

$$t^{-\gamma} L \qquad \text{and} \qquad t^{2-2\gamma} K \tag{27}$$

invariant.

Note that, in this way, we think of the two-parameter group (23) as a one-parameter (γ) family of one-parameter subgroups (26). Presumably, Eq. (24) becomes identical to Eq. (19) if and only if $a - 2b = 0$, that is, $a - 2\gamma a = 0$. This means that the subgroup $\gamma = \frac{1}{2}$ is an exact symmetry of the freely decaying turbulence. In other words, the larger symmetry (23) for infinite Reynolds number breaks down to its subgroup $\gamma = \frac{1}{2}$ for finite Reynolds numbers.

Each symmetry (26) is essentially time evolution. Following the arguments of Sec. IV we conclude that at adequately late times

$$L = \text{const}\; t^{\gamma} \qquad \text{and} \qquad K = \text{const}\; t^{2\gamma - 2}. \tag{28}$$

Thus, we have obtained certain power laws for the late behavior of the length scale and kinetic energy in freely decaying turbulence. The law for the dissipation rate ε follows immediately from Eq. (20),

$$\varepsilon = \text{const}\, t^{2\gamma - 3}. \tag{29}$$

By use of Eq. (17), then, the law for dissipation length L_ε turns out to be consistent with that of L, as it should. The law for Re_L also follows from (17):

$$\mathrm{Re}_L = \text{const}\, t^{2\gamma - 1}. \tag{30}$$

Summarizing, each value of γ defines a subgroup of the full symmetry group (23) for high Reynolds. Given a γ the time dependence of various quantities takes the form of specific power laws. *A priori*, not fixed without additional conditions, the exponent γ may be given an additional physical interpretation. Assume that for low wave numbers k the spectrum $E(k)$ is of the form

$$E(k) = Ck^{\sigma} + o(k^{\sigma}), \tag{31}$$

for some constants C and σ. Given the dimension of $E(k)$ and k and the constancy of C, this relation is invariant under (26) if and only if

$$\gamma = \frac{2}{\sigma + 3}. \tag{32}$$

That is, the subgroup (26) is fixed by the small wave-number behavior of the spectrum of the specific class of flows. It may be argued, see, e.g., Ref. [22], that C is actually constant as long as $1 \leqslant \sigma < 4$; moreover, the case $\sigma = 4$ holds marginally. That is, in those cases C is fixed by the initial conditions.

Decay exponents are usually expressed in terms of $n \equiv 2 - 2\gamma$, which is the kinetic energy decay exponent, $K \sim t^{-n}$. About the value of n there are well-known suggestions. They depend on the identification of C with quantities that are conserved under certain conditions. Kolmogorov [23] and Batchelor [24], based on the conservation of the Loitsyanky integral [25], derived $\gamma = \frac{2}{7}$, i.e., $n = \frac{10}{7}$. Saffman [26] set forth the hypothesis that the vorticity, and not the velocity correlator, is an analytic function in spectral space, by which he rediscovered the $\sigma = 2$ spectrum and Birkhoff's integral [27] and derived $\gamma = \frac{2}{5}$ i.e., $n = \frac{5}{6}$. Experimentally [28,29], both values of the decay exponent n are acceptable. The value $n = 1$ has also been suggested by other theoretical considerations for high but finite Reynolds numbers [30] and as the limiting value of the decay exponent for infinitely high Reynolds numbers [31–33]; this solution first appeared in Ref. [34].

The $n = 1$ decay solution for the finite Reynolds numbers of Ref. [30] can be obtained by recalling an observation given above that for finite Reynolds numbers the symmetry (23), essentially associated with infinite Reynolds numbers, breaks down to its $\gamma = \frac{1}{2}$ subgroup at finite Reynolds numbers, which means $n = 1$. Presumably, from Eq. (30), the Reynolds number is constant for this solution.

The $n = 1$ decay law may also be obtained in another way, which gives us the chance to make an additional comment on the analysis presented in this section. The Taylor microscale λ_g transforms as a length, the same as for r, according to the equation on the left in Eq. (21). This means that $\lambda_g = \text{constant}\, t^{\gamma}$. On the other hand, the equation on the right in Eq. (21) and the laws (28) and (29) imply $\lambda_g = \text{constant}\, t^{\frac{1}{2}}$. The reason there is no discrepancy is because regarding L as finite and the Reynolds number as virtually infinite for the symmetry (23) to hold means that λ_g is virtually zero. Put differently, if we want to think of the previous analysis as applying also to high but finite Reynolds numbers, then we must restrict ourselves to scales much larger than the Taylor microscale. It is then no accident that the power laws (28) can also be produced by models deriving from self-similarity of turbulence with respect to the integral scale L, as we shall discuss in Sec. VII. On the other hand, if we want finite Reynolds *and* to take into account scales of $O(\lambda_g)$ or less, then it must be $\gamma = \frac{1}{2}$, i.e., $n = 1$.

As it has such a direct impact on the arguments in this section, one may wonder why the $n = 1$ decay solution is not observed experimentally, even for the highest Reynolds numbers [equivalently, as $\gamma = \frac{1}{2}$ means $\sigma = 1$, a small wave-number spectrum $E(k) \propto k$ has not been verified]. The arguments possibly fail where one expects independence from the initial conditions. That expectation might hold the higher, but still finite, the Reynolds number is. This is why, in the best case, the $n = 1$ solution can possibly be regarded as describing well decaying turbulence for very high Reynolds numbers.

In the next two subsections we come to the problem of interest. The discussion parallels, in some sense, our previous remarks: Going from an infinitely high to any lower Reynolds number, the larger symmetry (23) breaks down, in this case, to its exact subgroup $\gamma = 0$ associated with the linearly forced turbulence, which is the exact symmetry (10) we started our discussion with. But, unlike the freely decaying case, in our problem a large length scale and a Reynolds number scale are *necessarily* present, eventually forcing the system toward the $\gamma = 0$ evolution. That amounts to reaching the stationary state.

C. Linearly forced isotropic turbulence

In this section we will think of our equations, including the symmetries and the associated power laws, with a constant explicitly added in the time coordinate t', as in Eq. (9), which we dropped for convenience since Sec. III.

Consider linearly forced turbulence in the description given by Eq. (8), which let us state, again:

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}' \cdot \nabla)\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \frac{\nu}{A(t' + t'_o)}\nabla^2 \mathbf{u}'.$$

The analog of the transformed Karman-Howarth equation (24) reads now

$$\frac{\partial}{\partial t'}\left(q_1'^2 f'\right) = \frac{1}{r^4}\frac{\partial}{\partial r}\left\{ r^4 \left[q_1'^3 h' + e^{-2\gamma a}\frac{\nu}{A(t'+t'_o)} 2 q_1'^2 \frac{\partial f'}{\partial r}\right]\right\}.$$

We find, again, that the subgroup $\gamma = 0$ of the group (23), i.e., the group (10) or, better, Eq. (9), is an exact symmetry of the linearly forced turbulence.

It is convenient to impose the condition that $t' = 0$ when $t = 0$. Consistent with this choice is to set $t'_o = A^{-1}$ and $t_o = 0$ in the basic transformation (6). The initial condition $u'(t' = 0)$ then is equal to $u_0 \equiv u(t = 0)$ by the transformation (3). Thus, we may simply refer to u_0 in either description. More generally, it is relevant to the late-time behavior that must be the same for an entire class of initial conditions to think in terms of initial conditions of $O(u_0)$ and constants $t'_o = O(A^{-1})$.

Consider a flow that starts off with velocities of order u_0 and a box of size l such that $u_0 \gg Al$. Equivalently, the turnover time is much smaller than forcing time scale A^{-1}, that is, $l/u_0 \ll A^{-1}$.

Given A, l, and ν there is a naturally defined Reynolds number in the problem:

$$\mathrm{Re}_A = \frac{Al^2}{\nu}. \tag{33}$$

That is, the condition $l/u_0 \ll A^{-1}$ can be rephrased such that the flow starts off with a very high Reynolds number, $\mathrm{Re}_L \gg \mathrm{Re}_A$.

Consider, then, times t' such that $l/u_0 \ll t' \ll t'_o = O(A^{-1})$. Looking at the Navier-Stokes equation above, we understand that for those times the turbulent flow is merely freely decaying with constant viscosity ν. If all previous inequalities hold strongly enough, then there will be time for the flow to evolve adequately toward its developed stage. This means that the quantities describing turbulence will evolve according to the power laws (28), (29), and (30), only now for primed quantities, and t is replaced by $t' + \text{constant}$. During this stage the Reynolds number decreases.

When t' becomes of the order A^{-1}, "linear forcing" kicks in. At this stage, the large Reynolds two-dimensional approximate symmetry (23) applied in the present variables produces evolution laws that are necessarily similar to those of the freely decaying case for the dimensionful quantities,

$$\begin{aligned} L' &= \text{const}\ t'^{\gamma}, \\ K' &= \text{const}\ t'^{2\gamma-2}, \\ \varepsilon' &= \text{const}\, t'^{2\gamma-3}, \end{aligned} \tag{34}$$

while the dimensionless Reynolds number has a different power law due to the time-varying ν':

$$\mathrm{Re}'_L = \text{const}\, t'^{2\gamma}. \tag{35}$$

The constant $t'_o = O(A^{-1})$ has been dropped for simplicity as, after all, these formulas hold when t' is relatively large. Now these equations are, at best, a very rough description of reality, but they give us some quantitative sense. (They are in a little better shape if Re_A is moderately large itself.) γ should vary and the state of the flow can be thought of as going through phases of different values of γ. Now the main difference with the freely decaying case is that the forcing itself lies "hidden" in the time-dependent viscosity. Therefore, these power laws, which were obtained by ignoring the viscosity term in the Navier-Stokes equation, cannot be true for various values of γ, as is the case in the freely decaying turbulence. If they were true, it would simply mean that the flow is still at stages where forcing is negligible. If the forcing and, therefore, the time-dependent viscosity term are present and operating, then we are left with a smaller symmetry in the problem. This is the subgroup $\gamma = 0$ of Eq. (23) that is an exact symmetry of the system. That is, the workings of the time-dependent viscosity term will eventually break down the larger symmetry to that smaller exact symmetry given by the $\gamma = 0$ subgroup and bring the system to the associated state, to which we now turn.

Let us first briefly summarize. There is a symmetry existing in the system for high Reynolds numbers, which, *a priori*, allows for arbitrary values of the parameter γ. This symmetry breaks down to its subgroup $\gamma = 0$ when the Reynolds number drops low enough. This is an exact symmetry of the system and, therefore, always holds. The final state must be the one that respects that exact symmetry. The power laws (34) and (35) imply that L' and Re'_L are constant, $\varepsilon' = \mathrm{const}\, t'^{-3}$ and $K' = \mathrm{const}\, t'^{-2}$. The transformations (15) and (18), the original variables of Eq. (1), show that everything, that is, K, ε, L_ε, and Re_L, is constant. We have reached the stationary state. K' and ε', which are time dependent in the description (8) of the problem, are related by Eq. (20), written for the primed quantities as follows:

$$\frac{dK'}{dt'} = -\varepsilon'. \tag{36}$$

The (34) and (35) power laws for $\gamma = 0$ and the transformations (15) translate that relation to

$$2AK = \varepsilon, \tag{37}$$

that is, we obtain the stationary state balance of energy production against dissipation.

D. Effects of the finite domain

In the previous subsection we saw the system reaching the stationary state, arguing as follows: In the decaying turbulence description (8), the forcing exists in the time-dependent viscosity. Therefore, one cannot draw even approximate conclusions about the late-time state from the large Reynolds symmetry (23) that ignores that viscosity term. The system should eventually conform only to the exact symmetry present in the problem and reach the state it induces; in the original variables this is the stationary state. Starting from high Reynolds numbers, one may then view the evolution of the system toward the stationary state as the breaking of a larger symmetry to a smaller symmetry. Here we revisit the argument from the point of view of the boundary conditions.

Note, first, a very interesting thing. Formally, from Eq. (32), the value $\gamma = 0$ corresponds to $\sigma = \infty$. This indicates that the power-law behavior of the spectrum $E(k)$ for small k degenerates and should be replaced by some other, much faster, decreasing law, perhaps some kind of exponential. That would be the behavior of a continuous spectrum applied in a physical system that lives in a finite region in space.

There is a good reason why we expect to see that. The system (1), or Eq. (8), is solved in a cubic domain, a fact we used in Sec. IV, but we did not imply there that we would not let its size become arbitrarily large or investigate what that would mean. We discuss now why an infinite size is meaningless or, put differently, that this limit possesses a meaning that differs entirely from the one expected. In linear forcing there is no intrinsic large length scale. Such a scale must be provided by a domain with a finite size. One may only then construct a scale for the Reynolds number at the late-time state under this forcing: This is given by the $\mathrm{Re}_A \equiv Al^2/\nu$ defined in Eq. (33). That is, in the linear forcing a domain size $l = \infty$ means we deal with the special state where $\mathrm{Re} = \infty$. This is certainly not the case in the freely decaying turbulence and neither is the case in the limited bandwidth forcing schemes. A different perspective is obtained if we think that large l essentially means a large Al compared with any specific initial condition u_0. An infinitely large l is then equivalent to initial conditions u_0 infinitely close to zero. This means that it will take forever to reach a state with an rms value of velocity of order Al. There will be no such thing as a late-time state, meaning, of course, a state established within a finite period of time. [A finite domain size means, presumably, that there are no wave numbers k between $O(l^{-1})$ and zero, therefore, any continuous approximation of the spectrum $E(k)$ must fall very rapidly for kl smaller than $O(1)$.] But perhaps the most straightforward way to understand the necessity of the finiteness of the domain is to revisit our previous arguments based on scaling symmetries.

We concluded earlier that linearly forced turbulence will eventually reach a state such that the Reynolds number Re_L and the integral length scale L are constant. [This holds in either description we have used due to the transformations (18).] This is now possible only if there are are available constants in the problem to set the scale for these quantities. As there is no large length scale in the dynamical statement of linearly forcing, as opposed to the limited bandwidth schemes, this is provided by the boundary conditions and, specifically, a necessarily finite domain size l. An infinitely large domain would simply mean that there would be no scale to set the scale of the constant L, that is, there would be no such thing as a stationary state. Observation through the DNS shows that, indeed, $L_\varepsilon \simeq l$, and, in fact, within a few percentage points of error [16] after a transient stage of evolution, L_ε was defined in Eq. (17). The existence of the domain size l implies, as we have already mentioned, a scale for the Reynolds number, the number Re_A. By $L_\varepsilon \simeq l$ one expects that $\mathrm{Re}_L \simeq \frac{1}{4}\mathrm{Re}_A$. Observation verifies this within a few percentage points [17]. These results are obtained only after time averaging the time series within the stationary state due to the relatively large fluctuations to which we will turn soon.

There is, of course, an inverse way to look at the issues discussed above. Let us take as given that the flow lives in a finite domain of size l. There is, then, a major implication following this fact. The presence of a fixed length l means that only the subgroup $b = 0$ (i.e., $\gamma = 0$) of the transformation (23) could possibly be a symmetry of the system, regardless of its dynamical equations. Thus, an alternative way to look at the evolution of the system toward a late-time state is that the latter is reached when the integral scale becomes comparable to the domain size l. The full group (23) then cannot even be a potential symmetry, even if it could be regarded as acceptable dynamically for adequately large Re_A. Its $\gamma = 0$ subgroup that is consistent with a fixed length will control the situation. Of course, the $\gamma = 0$ subgroup is an exact symmetry of the dynamical equations themselves, suggesting, from another point of view, that a finite domain is a most natural, if not necessary, thing in this system.

We conclude that the finiteness of the domain emerges as a crucial factor in understanding the flow evolving to a stationary state. The domain that we often refer to as the *box* now should be understood more carefully in terms of periodicity. This is what we discuss next.

VI. THE STATE OF ISOTROPY

In the previous section we restricted ourselves to flows obeying the conditions of homogeneity and isotropy. Let us review what that involves. Contract (1) with $\mathbf{u}$ and average. After a little rearranging we have

$$\frac{\partial K}{\partial t} = -\varepsilon + 2AK + \nu\nabla^2 K - \nabla\cdot\mathbf{J}, \qquad J_i \equiv \overline{\left(\frac{u^2}{2} + \frac{p}{\rho}\right)u_i}. \tag{38}$$

Homogeneity alone makes all locally defined correlators, such as K or J_i, independent of the position in space. That is, the last two terms in the last equation vanish. Of course, isotropy means homogeneity, so if the flow is assumed isotropic those two terms again go away. We are then left with

$$\dot{K} = -\varepsilon + 2AK. \tag{39}$$

In a stationary state we get $\varepsilon = 2AK$, whose form we already anticipated on dimensional grounds, deriving the final value (33) of the Reynolds number in terms of the box size l.

We should now recall that the box is a cubic domain together with periodic boundary conditions we impose on all fields. The periodic boundary conditions can be given some enlightening interpretations. One way to think of them is that we solve the Navier-Stokes equations in an infinite medium imposing periodicity l on the field $\mathbf{u}(x,y,z)$ in all three directions x, y, and z. That, in turn, means that homogeneity is not, *a priori*, broken: A box such that $\mathbf{u}(x,y,z)$ satisfies periodic boundary conditions can be drawn anywhere in the infinite medium. By the periodicity we apparently restrict ourselves to special kinds of flow such that there is an upper bound to the size of eddies or any spatially periodic structure in it.

Another way to think of the periodic boundary conditions is to interpret them as mere single-valued-ness of the fields while identifying the points of the boundary of the cube where the fields are supposed to be equal. This may be pictured if we go one dimension down. If we take a square and identify the opposite sides, we get a topological torus. A torus is a perfectly homogeneous space without boundary that is nontrivial globally and is not isotropic. Imposing periodic boundary conditions on the cube means we essentially solve Navier-Stokes equations on the three-dimensional analog of such a space, the three-torus.

An implication of periodicity and its peculiar nature is that an analog of Eq. (39) holds without assuming pointwise homogeneity of the flow. Let us denote by $\langle X\rangle$ the spatial average over the volume of the box of a quantity[1] with an ensemble average X. Averaging this way, we get an equation similar to Eq. (38) for box averages

$$\frac{d\langle K\rangle}{dt} = -\langle\varepsilon\rangle + 2A\langle K\rangle + \frac{1}{V_{\text{box}}}\int_{\text{bdy}}\{\nu\nabla K - \mathbf{J}\}\cdot d\mathbf{a}. \tag{40}$$

The last term is a surface integral over the boundary surface of the box. This integral is a sum of

$$\int_{x=l}(\nu\partial_x K - J_x)dydz - \int_{x=0}(\nu\partial_x K - J_x)dydz \tag{41}$$

plus two more analogous pairs of terms for the other two directions. As all fields are manufactured from correlations of the field $\mathbf{u}$ that satisfies periodicity $\mathbf{u}(x,y,z) = \mathbf{u}(x+l,y,z)$, and the same for the other two directions, any pair of terms such as (41) vanishes exactly and identically. In other words, even if pointwise homogeneity is not assumed, periodicity implies that

$$\frac{d\langle K\rangle}{dt} = -\langle\varepsilon\rangle + 2A\langle K\rangle \tag{42}$$

must hold exactly.

Alternatively, the vanishing of the boundary term in Eq. (40) follows automatically under the interpretation that we solve our problem on a three-torus: There simply is *no boundary*, as $x=0$ and $x=l$ describe the *same* surface somewhere on the three-torus. Whatever the interpretation of the boundary conditions, one ends up with (42) without assuming homogeneity of the flow.

This is a good thing to know. The box-averaged correlators $\langle X\rangle$ are the actual observables in the DNS. What is their relation to the ensemble averages X? Assuming that X are meaningful and under an ergodic hypothesis, X are represented by averaging $\langle X\rangle$ over suitable and adequately large intervals of time. That means, first, that although the motion of $\langle X\rangle$ is not the same as that of X it should nonetheless be bounded and appear as fluctuating around hypothetical stationary values. This is what is observed (Fig. 1). Those values should, of course, be the values of the correlators X.

Contemplating the far more complicated motion of the correlators $\langle X\rangle$ compared to that of X, one quickly realizes that both kinds of correlators obey the same basic dynamical equations. Thus, their difference lies somewhere else. The simplicity of the motion of X follows from their independence from the initial conditions of the flow and the symmetries of the dynamical equations and of the domain. Correlators $\langle X\rangle$ are not, *a priori*, independent of the initial conditions of the flow and none of the arguments of Sec. IV apply to them. Therefore, there are many more, and more complicated, solutions $\langle X\rangle$ than X.

In particular, in Sec. IV it was emphasized that the symmetries of the cube, combined with the symmetry (10), force a fair amount of isotropy on the solutions X. That will hold only on the average for the correlators $\langle X\rangle$. This is the actual result of numerical simulations (Fig. 2).

We consider the fluctuations of measures of isotropy important for the following reason. The "fair amount of isotropy" exhibited by X is not so harmless itself: Reasonably, any anisotropy is inherited by the correlators $\langle X\rangle$ and it

[1] That is, for example, $\langle K\rangle$ means simply $\langle\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\rangle$ not $\langle\frac{1}{2}\overline{\mathbf{u}\cdot\mathbf{u}}\rangle$. This is a little confusing but allows for a more compact notation.

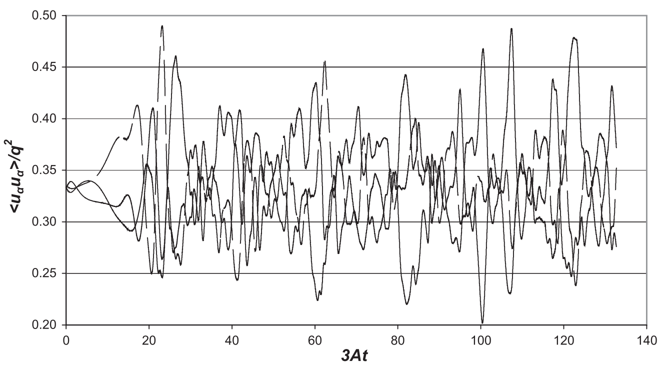

FIG. 2. Time evolution of the diagonal components of the normalized Reynolds stresses, $\langle u_\alpha u_\alpha \rangle / q^2$ (no sum) from the same run producing the time series in Fig. 1.

is enhanced in their description of turbulence. The origin of any enhancement of anisotropy is that in linear forcing there is no intrinsic large scale at which we feed the flow energy in some isotropic manner; the large scales are set by the domain itself and the scales comparable to its size are necessarily not isotropic. Subsequently, anisotropy is produced and maintained at those and smaller scales through forcing and cascade. Although the motion of the correlators X suggests that the system cannot be kicked off balance, the produced anisotropy will cause relatively large fluctuations of all quantities $\langle X \rangle$ describing turbulence.

In the next section we will try to lend some quantitative support to this intuitive picture. If deviations from isotropy are related to the fluctuations of all quantities, then any fluctuations should vanish in a perfectly isotropic setting. That may give us a sense of what happens when those fluctuations are continually generated. As we shall see the analysis is interesting in its own right as it reveals rather important dynamical properties of linearly forced turbulence.

VII. SELF-PRESERVING TURBULENCE AND STABILITY OF STATIONARITY

Studying the fluctuations around the stationary state is equivalent to studying the stability of that state as a fixed point of solutions, in the statistical sense. In fact, this is an alternative way to look at the main problem we have been concerned with, the stationary state as an attractor of solutions. Of course, such an analysis is a very difficult thing to do unless we resort to some suitable simplification.

According to the plan set at the end of the previous section, we shall assume that the flow is isotropic. Therefore, all correlators involved, which can be thought of either as ensemble correlators X or box averages $\langle X \rangle$, are assumed to have the properties required by that condition. We may then investigate the fate of any deviations away from the stationary state if the flow evolves remaining isotropic.

The evolution laws derived in the previous sections can be alternatively derived in isotropic turbulence from models relating the kinetic energy K and dissipation rate ε. These models can be deduced on dimensional grounds or, more systematically, by self-similarity arguments, which are fairly equivalent to the scaling arguments given here. The latter date to the work of von Karman and Howarth [35] and Batchelor [24]; see also Refs. [36,37]. One looks for self-similar solutions of the equations with respect to a single length scale $L(t)$, "self-preserving" turbulent flows. Assuming that the larger scales of the flow evolve in such a self-preserving manner, one chooses $L(t)$ to be the integral scale and one obtains a closed system of equations for the variables K and ε. That simple model can also be regarded as describing self-preserving turbulence of all scales but for infinitely high Reynolds numbers, essentially for inviscid flow.

One can straightforwardly apply the same arguments in the linearly forced turbulence. The spectral energy balance equation (22) becomes

$$\partial_t E(k) = -\partial_k T(k) - 2\nu k^2 E(k) + 2AE(k). \tag{43}$$

The origin of the additional term should be clear. The self-preserving development of the larger scales of the flow then implies, via standard steps that can be found in, e.g., Refs. [36,37], the model equation

$$\frac{d\varepsilon}{dt} = -C_\varepsilon^A \frac{\varepsilon^2}{K} + c_1 A\varepsilon, \tag{44}$$

where $c_1 = 3$ and C_ε^A is a dimensionless constant. Apart from the value of c_1, this equation could also have been guessed on dimensional grounds on requiring its right-hand side to be built from ε and K and A and be linear in A.

Integrating Eq. (43) over all wave numbers, we obtain, again, the exact equation (39), which we show here for convenience:

$$\frac{dK}{dt} = -\varepsilon + 2AK. \tag{45}$$

The system of Eqs. (45) and (44) is consistent with a static solution only for $2C_\varepsilon^A = c_1$. The special case of $C_\varepsilon^A = c_1/2 = 3/2$, predicted by large-scale self-preservation, implies that $L_\varepsilon = \text{const}$ at all times the model holds. This is consistent with the general idea regarding it. In Sec. VIII the large-scale self-preservation model [Eq. (44)] and the value $C_\varepsilon^A = 3/2$ will emerge again from a different perspective. The model can be easily solved exactly and indeed predicts that the flow approaches stationarity exponentially fast for all $C_\varepsilon^A > 1$ (the case $C_\varepsilon^A = 1$ is trivially consistent with stationarity).

A more elaborate analysis of the evolution of isotropic turbulence was presented in Refs. [30,31,38,39]. In those works, the self-similarity hypothesis is applied at the viscous equations of the flow, i.e., self-preservation is required to be true for all scales of turbulence for finite Reynolds. In the terminology of Ref. [30], self-preservation is complete. An implication of this requirement is that the self-similarity scale is the Taylor microscale λ_g.

From the point of view of linearly forced turbulence, all that sounds very relevant and interesting for the following reasons. First, the linearly forced turbulence comes to the intelligible part of its course when its Reynolds number approaches the value (33), which need not be very high at all; second, energy is generated uniformly at all points in the domain and all scales play a role in approaching or maintaining stationarity; and third, in this problem there is a natural scale for the Taylor length λ_g. It is the scale at which energy production balances dissipation in spectral space, as can be seen by Eq. (43):

$$\lambda_A = \sqrt{\frac{\nu}{A}}. \tag{46}$$

This is designated as a Taylor microscale because the stationary state value of the Taylor microscale, λ_{gs}, is of that order:

$$\lambda_{gs} = \sqrt{5}\lambda_A. \tag{47}$$

This follows from the definition (21) of λ_g and the stationary state total balance of energy production balances dissipation, $2AK = \varepsilon$. For these reasons, the Taylor microscale may be regarded as playing a particularly significant role in the dynamical aspects of linear forcing, perhaps more significant than in the freely decaying case. [On the other hand, as everything turns out to be approaching constancy, eventually the integral scale might be used as a self-similarity scale, a choice associated with the model (44), providing a more crude and late-time description of the evolution of the system.] In any case, this choice provides a closed two-equation model with some interesting properties.

We may then proceed as follows. There is another equation that we may use along with Eq. (45). One way to derive it is to start from the Karman-Howarth equation for linearly forced isotropic turbulence:

$$\frac{\partial}{\partial t}(q_1^2 f) = \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\left(q_1^3\, h + 2\nu\, q_1^2\,\frac{\partial f}{\partial r}\right)\right] + 2A\, q_1^2 f, \tag{48}$$

applying definitions (50) below. [The spectral energy balance equation (43) is a Fourier transform of Eq. (19).] Alternatively, and more instructively, we can do everything from scratch by differentiating ε with respect to time using its very definition as an ensemble or box-average correlator. Employing the Navier-Stokes equation (1) and applying the condition of isotropy on any arising correlator, one then arrives at

$$\frac{d\varepsilon}{dt} = \frac{7|S|}{3\sqrt{15\nu}}\,\varepsilon^{3/2} - \frac{7G}{15}\frac{\varepsilon^2}{K} + 2A\varepsilon, \tag{49}$$

where S (the velocity gradient distribution skewness) and G are defined by

$$S = \lambda_g^3 \frac{\partial^3 h}{\partial r^3}\bigg|_{r=0}, \qquad G = \lambda_g^4 \frac{\partial^4 f}{\partial r^4}\bigg|_{r=0}, \tag{50}$$

where f and h are the two-point double and triple point correlations of the velocity defined in Sec. V A. Equation (49) can also be derived by multiplying Eq. (43) by $2\nu k^2$ and using formulas equivalent to Eqs. (50) and (21) in wave-number space.

The system of Eqs. (45) and (49) is not closed, and the dependence of S and G on K and ε is unknown. Assume now that at some moment t_0 the flow becomes self-similar with a (time-dependent) similarity scale λ_0. This means f and g are functions of the dimensionless coordinate r/λ_0 alone, modulo a possible dependence on the initial conditions at t_0. Equation (21) now tells us that λ_0/λ_g must be a *constant*, depending only on the initial conditions at t_0. Thus, the similarity scale is indeed the Taylor microscale. From Eq. (50) we then have that S and G are constant and equal to the values they have at t_0: $S = S_0$ and $G = G_0$. The system of Eqs. (45) and (49) is now closed and we may study it.

Let us denote the stationary state values of the dissipation rate and kinetic energy by ε_s and K_s. Of course, they are related by $\varepsilon_s = 2AK_s$. We study the stability properties of $\varepsilon = \varepsilon_s$ and $K = K_s$ as a complete self-preserving solution of the system of Eqs. (45) and (49).

It will be convenient to define the quantity

$$g \equiv \frac{7G_0}{15}. \tag{51}$$

First, Eq. (49) implies that

$$\varepsilon_s^{1/2} = 2A\frac{3\sqrt{15\nu}}{7|S_0|}(g-1), \tag{52}$$

which implies that

$$g > 1.$$

It is useful to relate the value of g to the Taylor-scale Reynolds number $\mathrm{Re}_\lambda = (\frac{20}{3}\mathrm{Re}_L)^{\frac{1}{2}}$. From Eq. (17) we find that its stationary value $\mathrm{Re}_{\lambda s}$ reads

$$\mathrm{Re}_{\lambda s} = \frac{30}{7|S_0|}(g-1). \tag{53}$$

Define now small fluctuations ξ and ζ of ε and K around their stationary values:

$$\varepsilon = \varepsilon_s(1+\xi), \quad K = K_s(1+\zeta). \tag{54}$$

Inserting these expressions into Eqs. (45) and (49) and keeping only linear terms we obtain the following system of equations:

$$\begin{aligned}\frac{d\xi}{dt} &= -A(1+g)\,\xi + 2Ag\,\zeta,\\ \frac{d\zeta}{dt} &= -2A\,\xi + 2A\,\zeta.\end{aligned} \tag{55}$$

Its eigenvalues Γ read

$$\Gamma = \tfrac{1}{2}A[-(g-1) \pm \sqrt{(g-1)(g-9)}]. \tag{56}$$

By $g > 1$ we see that the real part of both eigenvalues is always negative. Fluctuations around the stationary state die out exponentially fast. That is, *modulo finite domain effects, the stationary state is stable as a complete self-preserving isotropic solution*. We may also view this result as providing further evidence that the stationary state is the natural final state of the linearly forced turbulence. [Presumably, one may observe that the exponentially fast approach to the stationary state is also the prediction of the simpler model (44).]

The previous analysis can be alternatively understood as follows. In order to derive the previous results we have assumed perfect isotropy. A reasonable assumption about the deviations from isotropy is that they originate from scales of order l. This means that, according to our conclusions in the previous section, the same can be said about the fluctuations around the stationary state. That is, one may attribute the generation of fluctuations to the interaction of the larger eddies with the periodicity, i.e., the restriction to their size. Through both forcing and cascade, fluctuations then are generated at all scales from l down to a certain scale where isotropy becomes a good approximation. There things differ. We may define correlators as spatial averages $\langle X\rangle_V$ over volumes V smaller than that maximum isotropic scale, i.e., within these volumes turbulence is isotropic (meaning homogeneity as well) to a good approximation, and then K and ε, understood as spatial averages $\langle X\rangle_V$, obey similar equations to those studied above. The entire previous analysis then works. This means, finally, that at adequately small scales the fluctuations are strongly suppressed but at all higher scales are maintained through forcing and cascade. The maximum isotropic scale should be (very) roughly related to the characteristic Taylor microscale of linear forcing $\lambda_A = (\nu/A)^{\frac{1}{2}}$, as below that scale the process of energy dissipation becomes stronger than energy production.

We may investigate the linear system (55) a bit further. Though this system is meant to serve us mainly for qualitative considerations, regarding the stability of the constant solution $K = K_s$ and $\varepsilon = \varepsilon_s$, there are some interesting remarks to be made about its solutions on the quantitative side. In the range $1 < g < 9$ the eigenvalues Γ are complex numbers. If we take for definiteness $|S_0| = 0.5$, this means that when $\mathrm{Re}_\lambda < 69$ the fluctuations are damped oscillations. [Presumably, this emergence of oscillations is a qualitative difference between the complete self-preservation model and the simpler model (44).] Inserting the solutions $\zeta = \zeta_0 e^{\Gamma t}$ and $\xi = \xi_0 e^{\Gamma t}$ for positive frequency into any of Eqs. (55) we obtain the phase difference and the relative amplitude of ε and K:

$$\xi_0 = \sqrt{g}\, e^{-i\phi}\, \zeta_0, \tag{57}$$

where ϕ is given by

$$\tan\phi = \frac{\sqrt{(g-1)(9-g)}}{g+3}. \tag{58}$$

As expected, the dissipation ε evolves with a phase delay with respect to the kinetic energy K and the energy production $2AK$. This corresponds to a time delay $\phi/|\mathrm{Im}\Gamma|$. The period of these damped oscillations is, of course, $2\pi/|\mathrm{Im}\Gamma|$.

In Fig. 1 we plotted the energy production $2AK$ and dissipation rate ε against time in units of $(3A)^{-1}$. Let us now shift the evolution of dissipation by one unit of time to offset its delay. The result is given in Fig. 3. One observes that, after that shift, the complicated oscillations appear in phase to a considerable degree of accuracy. Curiously, the time delay $\phi/|\mathrm{Im}\Gamma|$ in units of $(3A)^{-1}$ decreases from 1.5 to 0.5 in the range $1 < g < 9$. Moreover, the period $2\pi/|\mathrm{Im}\Gamma|$ is roughly an order of magnitude higher than $(3A)^{-1}$ for most values of

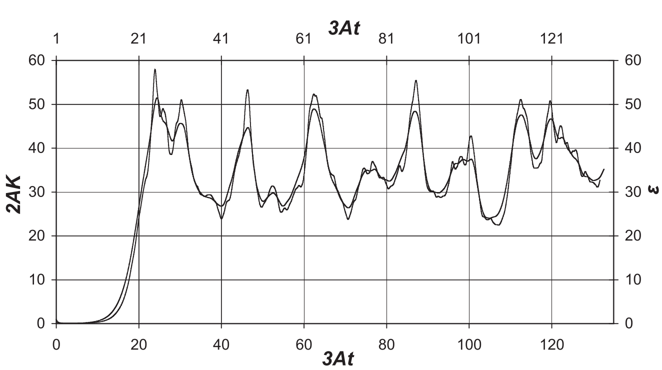

FIG. 3. The time evolution of the dissipation rate ε, shown in Fig. 1, is shifted in this figure by one unit of dimensionless time $3At$.

g, which as a number is not in disagreement with the picture in Fig. 3. Given that these numbers derive from a model that does not interact with the source of the fluctuations, it seems interesting that the oscillations it implies may encapsulate certain features of the actual fluctuation. There certainly is no identification between the actual fluctuations and those oscillations. For example, when $g \sim 9$, that is, $\mathrm{Re}_{\lambda s} \sim 70$, the damped oscillations are replaced by a purely decaying exponential, a qualitative change in the behavior that cannot be traced in the DNS results of Refs. [14,16,17].

The previous remarks derive from the quantitative characteristics of small fluctuations, and we may have overextended the applicability of the related formulas. Arbitrary fluctuations are described by the solutions of the full nonlinear model [Eqs. (45) and (49)]. This needs to be solved numerically. In terms of the dimensionless (hatted) kinetic energy, dissipation rate and time, defined, respectively, by $K = K_s\hat{K}$, $\varepsilon = \varepsilon_s\hat{\varepsilon}$, and $\hat{t} = 2At$, the nonlinear model reads

$$\frac{d\hat{K}}{d\hat{t}} = -\hat{\varepsilon} + \hat{K}, \qquad \frac{d\hat{\varepsilon}}{d\hat{t}} = (g-1)\,\hat{\varepsilon}^{3/2} - g\frac{\hat{\varepsilon}^2}{\hat{K}} + \hat{\varepsilon}. \tag{59}$$

The parameter g is related to $\mathrm{Re}_{\lambda s}$ by Eq. (53) and we again take for definiteness $|S_0| = 0.5$.

The system (59) is solved using the software MATHEMATICA. We consider a few specific cases. First, the difference of the initial conditions from the stationary state values is such to imitate the size of the observed fluctuations. This is shown in Fig. 4(a). Second, the kinetic energy $\hat{K}$ and dissipation rate $\hat{\varepsilon}$ start off from very close to zero, shown in Fig. 4(b). For those two cases we have chosen an adequately small Reynolds number so oscillations are visible. Finally, we consider the effect of higher Reynolds numbers. An evolution of $\hat{K}$ and $\hat{\varepsilon}$ for $\mathrm{Re}_{\lambda s} \sim 85$ is shown in Fig. 4(c).

The result is that the picture does not differ qualitatively from the one obtained from the small fluctuations. In Figs. 4(a) and 4(b), the time delay of the dissipation relative to the kinetic energy and the period of the damping appear essentially as predicted previously, and the oscillations of the dissipation are consistently larger as implied by Eq. (57). On the other hand, Fig. 4(b) shows a particular behavior of nonlinear solutions: If the initial condition is far away from the stationary state values the system undergoes large fluctuations before settling to those values. Figure 4(c) shows that by increasing the Reynolds number, any wiggling of the curves due to oscillatory behavior diminishes to extinction, which is, again, what we expected.

VIII. SUMMARY AND DISCUSSION

Direct numerical simulations of turbulence that is forced under the linear forcing scheme show a stationary late-time state that is not entirely expected. The stationary state is essentially quasistationary: all quantities have relatively large fluctuations though their time average can be predicted fairly well. In the present work we have attempted to understand how these phenomena are rooted in the properties of the system. The first revealing thing about it is that it can be transformed to a system of *decaying turbulence with decreasing viscosity* $\nu' = \nu(At')^{-1}$. The transformation involving a certain

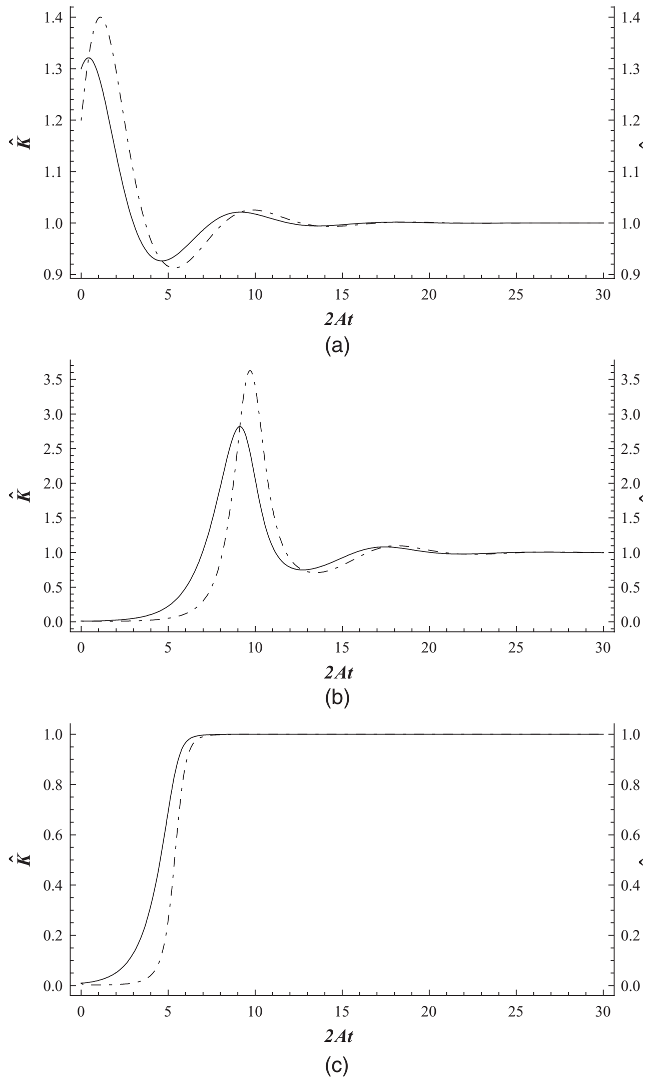

FIG. 4. Time evolution of the kinetic energy (solid line) and dissipation (dashed line) normalized by their stationary state values from numerical solutions of the system (59). (a) Initial conditions $\hat{K}(0) = 1.3$ and $\hat{\varepsilon}(0) = 1.2$ for $g = 2.2$, that is, $\mathrm{Re}_{\lambda s} \sim 10$. (b) $\hat{K}(0) = 0.01$ and $\hat{\varepsilon}(0) = 0.01$ for the same value of g. (c) $\hat{K}(0) = 0.01$ and $\hat{\varepsilon}(0) = 0.01$ for $g = 9.9$, that is, $\mathrm{Re}_{\lambda s} \sim 85$.

transformation of time $t \to t'$ and a subsequent transformation of all other quantities. One obtains, then, nothing but an equivalent description of the original system, and one can transform it back to the original description by given transformation rules.[2] That "new" system is interesting in its own right. One may show that at late times it evolves to a state where the total turbulent energy decays with time but the Reynolds number and integral length scale are constant, which may be shown quite elegantly. The formal similarity between the "new" system and the freely decaying turbulence allows us to use scaling symmetry considerations familiar in the latter when studying late-time evolution laws; see, e.g., Ref. [21] and references therein. [Needless to say, all these conclusions always refer to "late times," or a fully developed state of turbulence, in order for the system to have evolved away from the initial conditions adequately. This is necessary for the correlators to behave according to the general properties of the dynamical equations and the boundary conditions (or, perhaps, on the class of initial conditions) and not according to very particular initial conditions.]

First, if we drop the viscosity term, time dependent or not, the isotropic turbulence equations possess a two-dimensional scaling symmetry. From that symmetry one derives evolution for virtually any correlator one wants; the results are the well-known power laws of the freely decaying turbulence [21]. These power laws are not invariant under the full symmetry. They are determined when a single exponent is determined; that exponent fixes the one-dimensional subgroup of full symmetry under which they are invariant. That subgroup, i.e., the specific exponents of the power laws, are picked by the boundary and/or the type of the initial conditions and not by the dynamical equations. The subtle thing is that these results literary hold for the the Euler equations, which are not a well-defined limit of the Navier-Stokes for Reynolds numbers approaching infinity. Nonetheless, the results agree with observation even for moderately large Reynolds numbers. By the empirical fact that things work, the two-dimensional symmetry group can be regarded as an approximate symmetry for large but finite Reynolds numbers.

Second, when the viscosity term is not dropped, a case we must consider otherwise we restrict ourselves to situations such that the forcing is negligible, a single subgroup is picked, as an exact symmetry of the *viscous* equations. On the freely decaying side, the state specified this way was described by Ref. [30] and it is closely related to that described in Refs. [38,39]. It amounts to constant Reynolds number and kinetic energy decaying as t^{-1}. Though such a decay has not been observed, the state is of theoretical value. On the linearly forced side, in its description as decaying turbulence with decreasing viscosity $\nu' = \nu(At')^{-1}$, that state was mentioned above: Its Reynolds number and integral length scale are constant, and the kinetic energy decays as t'^{-2} and the dissipation as t'^{-3}. Now, though its freely decaying counterpart may be regarded nearly as unphysical, that state describes *stationarity in linear forcing*: If we transform back to the original variables, where there is an explicit force term, all quantities of interest are constant. The Reynolds number is of order $\mathrm{Re}_A = Al^2/\nu$, where l is the domain size. One case where the total evolution can be understood is one that starts from a much higher Reynolds number than Re_A. The settling of the system to stationarity can be understood as the breaking of the two-dimensional symmetry, which approximately holds at large Reynolds numbers, to its subgroup, which is an exact symmetry of the full viscous equations of isotropic turbulence.

It was mentioned above that in both descriptions of our problem, the integral scale L becomes constant in the late-time state. Lacking an intrinsic large scale at the level of the dynamical equations, linear forcing requires the scale to be set by the boundary conditions: Indeed, the domain size l sets the scale of the integral scale. For that reason, one cannot

[2]The viscosity ν' can never vanish: This may happen only for $t' = \infty$, where the transformation between the two descriptions becomes singular. Thus, the alternative description of our problem does not apply to questions involving the actual limit $t' \to \infty$. We do not need such a limit anywhere.

imagine linear forcing in an infinite domain, as one does with freely decaying or limited bandwidth forced turbulence. There would be no way to set the scale of the constant L at late times, thus there could no late-time state of stationarity, i.e., no way to approach such a state in a finite time. Conversely, given a finite domain, its size breaks the two-dimensional symmetry mentioned in the previous paragraph down to its subgroup that is in exact symmetry to the dynamical equations. Therefore, the late-time state of the system must be the stationary state described above. These facts underline the importance of the finite domain in this problem.

Implicit in our comments above is the assumption that turbulence is isotropic. The reason is threefold. First, it is far more convenient to analyze the simpler setting and realize what might change when there are deviations from it. Second, although the symmetries we have used hold at the level of the Navier-Stokes equations, which means that these symmetries are inherited by all dynamical equations for the correlators, it feels like an excessively strong assumption that our arguments, which involve the assumption of independence from the initial conditions, apply also to the general anisotropic turbulence. Third, there is a fair amount of *local* isotropy imposed by the solenoidal nature of the velocity field and the symmetries of the cubic domain, as discussed in Sec. IV. Nonetheless, the isotropy is broken at the domain size scale. The periodic boundary conditions make sure that the flow lives in an everywhere homogeneous space, a topological three-torus, but isotropy is broken at the scale of its size, as one can certainly distinguish directions at the scale of the domain size. We have mentioned already that linear forcing has no intrinsic large length scale and that the finiteness of that anisotropic domain is a major component of the forcing. Therefore, anisotropies are generated and, by cascade, transferred at all smaller scales down, perhaps, to the Taylor scale eddies, where dissipation wins over the forcing. This is our argument for the origin of the rather significant deviations from isotropy observed in direct numerical simulations; it is rooted in the otherwise necessary finiteness of the domain. Now, ignoring the effects of the large-scale anisotropies by assuming isotropic turbulence, we found a late-time state where everything was constant, i.e., we got statistical staticity, actually, rather than stationarity. That was obtained by working with the statistical ensemble-averaged correlators, which are assumed not to depend on initial conditions; that is the great simplification allowing us to proceed without dealing with all the details of the Navier-Stokes equations. The actual DNS observables, the box-averaged correlators, are actually fluctuating. It must be, then, that the mathematical condition of independence from the initial conditions imposed on the ensemble-averaged correlators is partially weakened as a physically viable condition by the large-scale breaking of isotropy in the domain. We then conclude that the rather significant fluctuations observed in the stationarity state in the numerical simulations are due to the large-scale anisotropies, i.e., the very finiteness of the domain in this problem.

The last argument is possibly strengthened, and we also gain some different understanding of our problem, if we proceed and simplify it to a soluble model. First, as mentioned above, the stationary state as appears in the description of decaying turbulence with decreasing viscosity is an analog of a freely decaying turbulence state described in Ref. [30]; see also references therein. That state was found by applying self-similarity of isotropic turbulence holding at all scales, called complete self-preservation of isotropic turbulence. That requires the similarity scale to be the Taylor microscale. Second, we observe that in the linear forcing there is a natural scale for the Taylor microscale, $\lambda_A = (\nu/A)^{\frac{1}{2}}$. This is, in fact, the only intrinsic length scale in the problem, i.e., without bringing in the finite domain. Complete self-preservation then sounds all the more relevant and attractive and we derive the corresponding model. It turns out that the model possesses a single stable fixed point. Therefore, assuming isotropy and reducing the unknowns by self-similarity, the stationary state is found again as the late-time state of our system, in the sense of a state where everything is constant. One way to interpret this result, is to say that is provides more evidence that isotropy forces any deviations from stationarity to vanish. Of course, one may attribute that to the effect of self-similarity reduction, and we have no good argument against such an objection. Nonetheless, the model is indeed unaware of the anisotropies generated at the larger scales and is consistent with our conclusions based on the symmetry arguments. There are some additional properties of the linear forced turbulence associated with the effects of the finite domain as well as with its formal affinity to freely decaying turbulence, which we may briefly discuss here.

Denote by Δu_l the longitudinal velocity difference. The second- and third-order structure functions are related to the correlation functions f and h, introduced in Sec. V, by $\overline{(\Delta u_l)^2} = 2q_1^2(1-f)$ and $\overline{(\Delta u_l)^3} = 6q_1^3 h$. For adequately high Reynolds numbers there is a range of distances (the inertial range) where $\overline{(\Delta u_l)^2} = C_2(\varepsilon r)^{2/3}$ and where C_2 is a constant. Consider, first, decaying turbulence. It evolves according to the power laws (28), and the integral scale is proportional to t^γ. The law for ε can be deduced. It is then straightforward to show that they satisfy the $K-\varepsilon$ model equation [Eq. (44)] for

$$C_\varepsilon = \frac{3-2\gamma}{2-2\gamma}, \tag{60}$$

and, of course, $A = 0$. Using the Karman-Howarth equation (19) it is then straightforward to show [40,41] that for *very high but finite* Reynolds numbers, and within the inertial range [more specifically as long as r/λ_g is a number of $O(1)$], the two-thirds law of the second-order structure function implies specific finite Reynolds number corrections to the four-fifths law of the third-order structure function, of $O(\mathrm{Re}_\lambda^{-2/3})$. The result is [40,41]

$$\overline{(\Delta u_l)^3} = -\frac{4}{5}\varepsilon r \times \left[1 - \frac{5\times 15^{\frac{2}{3}}}{17} C_\varepsilon C_2 \mathrm{Re}_\lambda^{-\frac{2}{3}} \left(\frac{r}{\lambda_g}\right)^{\frac{2}{3}} - \left(\frac{25}{3}\right)^{\frac{1}{3}} C_2 \mathrm{Re}_\lambda^{-\frac{2}{3}} \left(\frac{r}{\lambda_g}\right)^{-\frac{4}{3}}\right]. \tag{61}$$

Consider the same question in the linearly forced turbulence. One may follow the same steps, starting from the Karman-Howarth equation with linear forcing, Eq. (48). One finds a result entirely similar to Eq. (61) on replacing

$$C_\varepsilon \to -\frac{K}{\varepsilon^2}\frac{d\varepsilon}{dt} + \frac{3AK}{\varepsilon}. \tag{62}$$

Observe now that if we think of the right-hand side of this substitution as a constant, then we rediscover the model equation (44); the constant is what we denoted there by C_ε^A. Equation (44) is derived assuming self-similarity (self-preservation) of the larger scales of turbulence with respect to the integral scale L for high Reynolds numbers in both the linearly forced ($A \neq 0$) and freely decaying case ($A = 0$). In all, by self-preservation we obtain a similar result of the form (61) in both kinds of turbulence, differing only in the value of the constants C_ε^A and C_ε. On the linearly forced side, self-preservation requires $C_\varepsilon^A = 3/2$ and Eqs. (44) and (45) require that $L = \text{const}$. At first sight there is no such restriction on the freely decaying side. In all, there appears to be a correspondence between linearly forced and freely decaying turbulence, though this correspondence appears inexact.

If we require $C_\varepsilon^A = C_\varepsilon$, then from Eq. (60) we have that $\gamma = 0$. In other words, if the decaying turbulence evolves according to $L \sim \text{const}$ (and $K \sim t^{-2}$), then its structure function expression (61) is exactly similar to that of the linearly forced turbulence. That is, the correspondence between the two flows can be exact.

The $K \sim t^{-2}$ evolution is too fast compared to the usually observed decay laws, discussed in Sec. V B. Such power laws can be reproduced if we choose the constant C_ε to differ from $3/2$, a fact regarded as an imperfection of the correspondence in Ref. [15], where it was first pointed out. On the other hand, the origin and the nature of the correspondence seem to have been overlooked in Ref. [15].

The key role is played again by the finiteness of the domain. As emphasized in Sec. V D and just above a container is a necessary thing when turbulence is linearly forced. It is, therefore, not much of a surprise that similarities between linearly forced and freely decaying turbulence are more detailed when the decaying side evolves in a way consistent with the existence of a container: For adequately high Reynolds numbers that means $L \sim \text{const}$ {and the rest of the power laws [Eqs. (28)–(30)] for $\gamma = 0$}. The mathematics of self-similarity of turbulence with respect to the scale L then imply exactly the same formula [Eq. (61)] for both kinds of turbulence.

The next obvious question is as follows: What kind of modifications does linear forcing need in order to reproduce aspects of a generic decaying turbulence, associated with Eq. (60) and an evolution law $L \sim t^\gamma$? Two immediate guesses are to consider a time-dependent rate $A = A(t)$ or to consider a time-dependent box whose size l evolves according to $l \sim t^\gamma$. The analysis of such possibilities is left for future work.

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