Bull Eng Geol Environ
DOI 10.1007/s10064-011-0413-4

DISCUSSION

Steady state groundwater seepage in sloping unconfined aquifers RP Chapuis, *Bull Eng Geol Environ* 70:89–99. doi:10.1007/s10064-010-0282-2

Antonis D. Koussis · Evangelos Akylas

Received: 30 May 2011 / Accepted: 20 November 2011

This Commentary has three aims: (1) to state the complete extended Boussinesq equation, from which its abridged and commonly used form derives, and the condition under which the latter represents the complete one with sufficient accuracy; (2) to draw attention: (a) to an analytical steady-state solution of the nonlinear extended equation of Boussinesq derived by Henderson and Wooding (1964) and reworked by Basha and Maalouf (2005), and (b) to McEnroe's (1993) solution of Eq. 21 in Chapuis (2011); and (3) to discuss steady-state solutions of two linearised forms of the extended equation of Boussinesq, giving criteria under which the linear solutions approximate the nonlinear solutions well.

The complete form of the extended equation of Boussinesq

Subsurface flow on a sloping base (also called subsurface stormflow or hillslope flow) has been studied extensively. Henderson and Wooding (1964) and Wooding and Chapman (1966) laid its mathematical foundations based on the Dupuit-Forchheimer theory of unconfined flow (hydrostatic pressure and thus constant potential over the depth, H, measured normal to the bed) deriving the extended equation of Boussinesq that accounts for a sloping base. Wooding (1966) examined the accuracy of that *hydraulic* equation via application of conformal mapping. In the notation of Chapuis (2011), the soil has saturated hydraulic conductivity k_{sat} and specific yield f, rests on an impervious bed inclined against the horizontal at an angle α, and is recharged at a constant rate per unit horizontal area N. Then, the discharge per unit width (planar flow), at time t and location x', measured from the top of hill along the inclined base of length L, is given by (Henderson and Wooding 1964; Wooding and Chapman 1966; Childs 1971)

$$q(x,t) = Hk_{sat}\left(\sin\alpha - \frac{\partial H}{\partial x'}\cos\alpha\right), \tag{1}$$

where flow is properly positive in the $+x'$ direction. The storage balance equation is

$$f\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x'} = N\cos\alpha + N\frac{\partial H}{\partial x'}\sin\alpha. \tag{2}$$

The term $N(\partial H/\partial x')\sin\alpha$ on the right-hand side of Eq. 2 derives from the scalar product of the recharge vector and the unit normal of a free surface element, but is rarely included in the volume balance. Combining Eq. 1 with Eq. 2 yields the complete extended equation of Boussinesq for unconfined flow over an inclined base (Akylas et al. 2006):

$$f\frac{\partial H}{\partial t} + k_{sat}\sin\alpha\left(1 - \frac{N}{k_{sat}}\right)\frac{\partial H}{\partial x'} - k_{sat}\frac{\partial}{\partial x'}\left(H\frac{\partial H}{\partial x'}\right)\cos\alpha = N\cos\alpha. \tag{3}$$

Henderson and Wooding (1964) neglected the last term in Eq. 2, obtaining Eq. 3 with $(1 - N/k_{sat})$ replaced by 1. That equation has been adopted widely (e.g., Beven 1981;

A. D. Koussis (✉)
Institute for Environmental Research and Sustainable Development, National Observatory of Athens, I. Metaxa & Vas. Pavlou, Palea Penteli, 15452 Athens, Greece
e-mail: akoussis@noa.gr

E. Akylas
Department of Civil Engineering and Geomatics, Technical University of Cyprus, Limassol, Cyprus

Published online: 06 March 2012

Koussis and Lien 1982; Koussis 1992; Koussis et al. 1998; Verhoest and Troch 2000; Pauwels et al. 2002; Basha and Maalouf 2005) and holds for $N/k_{sat} \ll 1$ (e.g., for $N =$ 24 mm day^{-1} and $k_{sat} = 2.5$ m day^{-1}, $N/k_{sat} \approx 0.01$). Chapman (2005) gives an excellent review of the governing equations, also comparing the solution of Eq. 3 to that of a linearised form; he shows that the mean steady flow depths, computed with $(1 - N/k_{sat})$ and with that factor set to 1, vary appreciably only for large N/k_{sat} values, $N/k_{sat} \geq 0.1$, say. Given that typically $N/k_{sat} < 0.05$ and the uncertainty in the values of k_{sat} and N, the approximation seems justified in most cases. Unfortunately, Chapuis (2011) stated the extended Boussinesq equation for the case of constant discharge ($N = 0$) but considered recharge in his solution.

The analytical steady-state solution of the nonlinear extended equation of Boussinesq of Henderson and Wooding (1964)

Henderson and Wooding (1964) derived an analytical solution of the nonlinear extended equation of Boussinesq for steady-state flow with recharge, i.e., Eq. 3 with $\partial H/\partial t = 0$ (and also $N/k_{sat} \ll 1$). The resulting governing equation can be integrated analytically once, yielding, in essence, the flux, Eq. 1 (see Eq. 2 with $\partial H/\partial t = 0$):

$$H\left(\frac{\partial H}{\partial x'} - \frac{H\tan\alpha - Nx'/k_{sat}}{H}\right) = const. \tag{4}$$

Equation 4 is the point of departure for developing solutions that vary depending on the boundary conditions of the problem. In their discussion of such solutions, Henderson and Wooding (1964) considered the case of no-flow at the top of the hill, $x' = 0$, whence the constant on the right-hand side of Eq. 4 vanishes, leading to two possibilities (see Eq. 1): either $H = 0$, or $\partial H/\partial x' = \tan\alpha$ (i.e., the depth is finite but the free surface is horizontal there). In both instances the first-order ordinary differential equation 4 can be integrated. In the second case, the solutions, presented by Henderson and Wooding (1964) in non-dimensional form, depend on the value of the dimensionless source term $\lambda = 4N\cos\alpha/k_{sat}\sin^2\alpha$. Basha and Maalouf (2005), adopting the same form of λ, give solutions for $\lambda > 1$ and $\lambda \leq 1$, also considering a non-zero depth condition at the foot of the hill. These published solutions are not reproduced here. It should be also noted that Basha and Maalouf (2005) acknowledge that the commonly used form of the extended Boussinesq equation, which they themselves use, is an approximation of the complete one, Eq. 3.

McEnroe (1993) combined the steady-state volume conservation equation $q = Nx$ with the approximate discharge expression, elegantly derived by Chapman (1980), $q = -k_{sat}\cos^2\alpha\ h' d/dx\ [(h' + z)]$, where $z = (L - x)\tan\alpha$ the bed elevation, and obtained the equation

$$k_{sat}h'\frac{dh'}{dx}\cos^2\alpha - k_{sat}h'\tan\alpha\cos^2\alpha + Nx = 0 \tag{5a}$$

which, after introducing the coordinates' relationship $x = x'\cos\alpha$, writes

$$k_{sat}h'\frac{dh'}{dx'} - k_{sat}h'\sin\alpha + Nx' = 0 \tag{5b}$$

(McEnroe incorrectly used z as $x\tan\alpha$, but his final Eq. 9 is correct.) McEnroe solved Eq. 5b, which differs from Eq. 14 of Chapuis (2011) in that: (i) it lacks the term $Q_o/\cos\alpha$ (it assumes zero–flow at $x' = 0$) and—following Chapman (1980)—the first term does not have the $\cos^2\alpha$, which however is very nearly 1 for the slopes considered by Chapuis. Except for these differences (amounting to a shift in the transformed x' variable, X, of Chapuis), Eq. 21 of Chapuis and McEnroe's Eq. 9 (Eq. 5b here), and its dimensionless counterpart Eq. 10, are <u>formally</u> identical, with compatible solutions. McEnroe made his nondimensional Eq. 10 separable by changing variables to $X = x'/L$ and $u = h'/(x\ \tan\alpha)$ (equivalent to making Eq. 5b here separable in the variables x' and h'/x'), obtaining and solving the differential equation $-dX/X = udu/(R^* - u + u^2)$, where $R^* = N/k_{sat}\sin^2\alpha$.

Linearisation solutions of the extended Boussinesq equation

Two ways to linearise Eq. 3 have been advanced. The first assumes that the variation of the depth over the slope is small and implies that $(\partial H/\partial x)^2$ is small relative to the other terms in the equation and may be neglected, while $\partial/\partial x'(H\partial H/\partial x') \approx H_o\partial^2 H/\partial x'^2$. This is the basis of the linearised equation first introduced by Koussis and Lien (1982), and used extensively since then to model subsurface flow on a sloping base, in which H_o [$=pD$ in the notation of Chapuis (2011), his Eq. 3] must be determined, e.g., as proposed by Koussis (1992). We state that equation here, corrected to include the factor $(1 - N/k_{sat})$:

$$f\frac{\partial H}{\partial t} + k_{sat}\sin\alpha\left(1 - \frac{N}{k_{sat}}\right)\frac{\partial H}{\partial x'} - k_{sat}H_o\frac{\partial^2 H}{\partial x'^2}\cos\alpha = N\cos\alpha \tag{6}$$

Chapman (1995) proposed the *quadratic* [terminology of Basha and Maalouf (2005)] linearisation $\eta = H^2$, so that the linearised extended equation of Boussinesq writes:

$$f\frac{\partial \eta}{\partial t} + k_{sat}\sin\alpha\left(1 - \frac{N}{k_{sat}}\right)\frac{\partial \eta}{\partial x'} - k_{sat}\sqrt{\eta_o}\frac{\partial^2 \eta}{\partial x'^2}\cos\alpha = 2\sqrt{\eta_o}N\cos\alpha \tag{7}$$

Equations 6 and 7 are Linear Advection–Diffusion equations [the LAD model of Koussis (1992) and Akylas et al. (2006)] that are appropriate under different conditions. Equation 6 properly reproduces the kinematic wave behaviour in the small-depths limit, whence $H_o = 0$ (steep slopes, high k_{sat}, low N, i.e., small λ; see formal criterion below), while Eq. 7 recovers Dupuit's equation $k_{sat}d/dx(HdH/dx) = -N$ in the limit $\alpha = 0$. Each one of these equations, expectedly, fails in the limiting case where the other is exact: in the first limiting case the approximate second-derivative term $H_o\partial^2H/\partial x'^2$ is eliminated, leaving the kinematic wave equation, which is exact under the stated limiting conditions; in the second limiting case, $\alpha = 0$, the approximated first spatial derivative term is eliminated, leaving (with $\partial H/\partial t = 0$, at steady state) the Dupuit equation, which holds exactly for a horizontal base. Evidently, Eqs. 6 and 7 formally admit the same solutions, which must be translated according to the appropriate linearisation variable. Akylas et al. (2006) identified the dimensionless parameter $L\sin\alpha(1 - N/k_{sat})/H_o\cos\alpha \approx L\sin\alpha/H_o\cos\alpha = L\tan\alpha/H_o \approx \Delta Z/H_o$ as controlling the flow behaviour; it expresses the ratio of gravity–driven flow to gradient–driven flow (*hydraulic diffusion*). H_o ($=pD$) is a function of the dimensionless source term $\lambda = 4N\cos\alpha/k_{sat}\sin^2\alpha$ of Henderson and Wooding (Akylas et al. (2006) use $R = \lambda/4$), which shows that the slope influences the flow behaviour more than the hydraulic conductivity or the recharge rate, but not exclusively, so that in the same geologic medium the flow can be more kinematic or more diffusive depending on the recharge rate.

In their comprehensive analysis, Basha and Maalouf (2005) compared solutions to both linearised extended Boussinesq equations at steady state with the exact nonlinear steady-state solution (their Fig. 2): they found the first, "linear" solution (Koussis) suitable for small λ and the second, "quadratic" solution (Chapman) suitable for large λ (but did not demarcate the "small–large" λ-boundary), an expected behaviour according to the above analysis. Akylas et al. (2006) compared the analytical steady-state solution of the linearised Eq. 6, for $q(x' = 0) = 0$ and $H(x' = L) = 0$, to a numerical nonlinear steady-state solution (their Fig. 2) and found it to perform well up to $\lambda \approx 1.5$. Generally, the linearised Eq. 6 or 7 is readily solved for non-zero depth at $x' = L$ and non-zero flow at $x' = 0$; e.g., Pauwels et al. (2002) present a solution to the linearised Eq. 6 for a non-zero depth at the foot of the hillslope, as do Basha and Maalouf (2005) for flow at steady-state.

Concluding remarks

1. McEnroe (1993) was the first to derive and solve Eq. 14 of Chapuis (2011) (without the Q_o term).
2. We question Chapuis's preference for Eq. 14 and its implicit solution Eq. 21 to the also implicit and equally awkward to handle, but better-founded solution of Henderson and Wooding (1964) of the extended equation of Boussinesq. Assuming that the results of Eq. 21 are close to those of the rigorous solution of Henderson and Wooding, the former could be chosen if it were clearly simpler to handle than the latter, but this is not the case.
3. Furthermore, we are of the opinion that the two solutions of the linearised extended equation of Boussinesq, properly used, afford reliable estimates for practical work, given the usual uncertainty surrounding the schematised hillslope geometry, the values of the aquifer's parameter and the recharge. These linear solutions allow superposition as well as modelling of transient flow in sloping aquifers [Akylas and Koussis (2007); Koussis et al. (2007)].

References

Akylas E, Koussis AD (2007) Response of sloping unconfined aquifer to stage changes in adjacent stream I. Theoretical analysis and derivation of system response functions. J Hydrol 338:85–95

Akylas E, Koussis AD, Yannacopoulos AN (2006) Analytical solution of transient flow in a sloping soil layer with recharge. J Hydrol Sci 51(4):626–641

Basha HA, Maalouf SF (2005) Theoretical and conceptual models of subsurface hillslope flows. Water Resour Res 41:W07018. doi:10.1029/2004WR003769

Beven K (1981) Kinematic subsurface stormflow. Water Resour Res 17(5):1419–1424

Chapman TG (1980) Modeling groundwater flow over sloping beds. Water Resour Res 16(6):1114–1118

Chapman T (1995) Comment on: the unit response of groundwater outflow from a hillslope. Water Resour Res 31(9):2376–2777

Chapman TG (2005) Recharge-induced groundwater flow over a plane sloping bed: solutions for steady and transient flow using physical and numerical models. Water Resour Res 41:W07027. doi:10.1029/2004WR003606

Chapuis RP (2011) Steady state groundwater seepage in sloping unconfined aquifers. Bull Eng Geol Environ 70:89–99. doi:10.1007/s10064-010-0282-2

Childs EC (1971) Drainage of groundwater resting on a sloping bed. Water Resour Res 7(5):1256–1263

Henderson FM, Wooding RA (1964) Overland flow and groundwater flow from a steady rainfall of finite duration. J Geophys Res 69(8):1531–1540

Koussis AD (1992) A linear conceptual subsurface storm flow model. Water Resour Res 28(4):1047–1052

Koussis AD, Lien L-T (1982) Linear theory of subsurface stormflow. Water Resour Res 18(6):1738–1740

Koussis AD, Smith ME, Akylas E, Tombrou M (1998) Groundwater drainage flow in a soil layer resting on an inclined leaky bed. Water Resour Res 34(11):2879–2887

Koussis AD, Akylas E, Mazi K (2007) Response of sloping unconfined aquifer to stage changes in adjacent stream II. Applications. J Hydrol 338:73–84

McEnroe BM (1993) Maximum saturated depth over landfill liner. J Env Eng 119(2):262–270

Pauwels VRN, Verhoest NEC and De Troch FP (2002) A metahillslope model based on an analytical solution to a linearized Boussinesq equation for temporally variable recharge rates. Water Resour Res 38(12):33. doi:10.1029/2001WR000714

Verhoest NEC, Troch PA (2000) Some analytical solutions of the linearized Boussinesq equation with recharge for a sloping aquifer. Water Resour Res 36(3):793–800

Wooding RA (1966) Groundwater flow over a sloping impermeable layer, 2. Exact solutions by conformal mapping. J Geophys Res 7(12):2903–2910

Wooding RA, Chapman TG (1966) Groundwater flow over a sloping impermeable layer, 1. Application of the Dupuit-Forchheimer assumption. J Geophys Res 71(12):2895–2902