ground water

Methods Note/

Slug Test in Confined Aquifers, the Over-Damped Case: Quasi-Steady Flow Analysis

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Abstract

In the great majority of slug tests performed in wells fully penetrating confined geologic formations, and for over-damped conditions, the response data are evaluated with the transient-flow model of Cooper et al. (1967) when the radial hydraulic conductivity K_r and the coefficient of specific storage S_s are to be estimated. That particular analytical solution, however, is computationally involved and awkward to use. Thus, groundwater professionals often use a few pre-prepared *type-curves* to fit the data by a rough matching procedure, visually or computationally. On the other hand, the method of Hvorslev (1951), which assumes the flow to be quasi-steady, is much simpler but yields only K_r estimates. In this work, we develop a complete quasi-steady flow model that includes a storage balance inside the aquifer and allows estimating K_r and S_s simultaneously, through matching of the well response data to a type-curve. The new model approximates the model of Cooper et al. closely and has the practical advantage that its solution type-curves are generated easily using an electronic spreadsheet, so that the optimal fit of data by a type-curve can be readily automated.

Scope of Work

Slug tests offer a fast and inexpensive means of estimating the hydraulic parameters of a geologic formation, and are very well suited for contaminated site assessment because no water is essentially withdrawn. In this in situ test, head variations are generated in the aquifer through a rapid change of the water level in the borehole, induced by adding or *bailing* water, by placing a metal piece (a *slug*) in the well casing causing a change of the water level, or pneumatically. Several methods have been developed for evaluating the head variation observed in the well (response), in order to estimate the hydraulic parameters of the geologic formation across the screened or open section of the test well (Butler 1998). The method of Hvorslev (1951) is the simplest of these, but allows estimating only the geologic formation's hydraulic conductivity. In contrast, the method of Cooper et al. (1967) permits estimating also the specific storage, but is quite involved mathematically. Chirlin (1989) has analyzed incisively the physics underpinning these two methods. The monograph of Butler (1998) is an excellent source of knowledge regarding theoretical and practical aspects of the slug test.

In this Methods Note we consider slug tests performed in homogeneous confined aquifers fully penetrated by the test well (Figure 1), intending to show that the quasi-steady flow model, on which the method of Hvorslev is founded, is not inherently limited to the estimation of a formation's hydraulic conductivity, but can be extended to estimate a formation's specific storage coefficient as well. In a quasi-steady solution, a series of steady states is substituted for the transient process, that is, the evolution of the physical system is considered to take place in abrupt steps from one steady state to the next one. This concept is well established in subsurface hydrology, dating back to Lembke (1886, 1887), and is

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still in use (e.g., the quasi-steady solution of Verhoest and Troch (2000) and its assessment by Akylas et al. (2006)).

Slug test models may be broadly subdivided in those that solve the governing flow (*field*) equation over the entire domain and in those that focus on the head variation inside the well, relating it to the flux across the well screen that prevails throughout the aquifer at each instant, according to the quasi-steady flow approximation. The model of Cooper et al. (1967) belongs to the first category and the model of Hvorslev (1951) to the second. The second model type cannot yield an estimate of a formation's specific storage coefficient, because it does not invoke a storage balance inside the aquifer that would account for storage changes. However, the quasi-steady flow model can serve to set up a storage balance for the aquifer. Here, we develop such a first-category model based on the approximate, but complete representation of quasi-steady flow dynamics, and devise a method for slug test data evaluation that allows estimating K_r and S_s.

Theoretical Foundation: The Model of Cooper et al. (1967)

The equation governing groundwater flow induced by slugging a well fully penetrating a homogeneous confined aquifer of constant thickness, Figure 1, is derived from the law of mass conservation $\nabla \cdot (\rho \boldsymbol{q}) = \partial(\rho n)/\partial t$, with the velocity $\boldsymbol{q}$ expressed by Darcy's law and introducing constitutive laws for the water and the aquifer, leading to $\nabla \cdot \boldsymbol{q} = S_s \partial h/\partial t$ (de Marsily 1986). In polar coordinates (rotational symmetry; no vertical variations) this governing equation is

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S_s}{K_r}\frac{\partial h}{\partial t}, r > r_s, \tag{1}$$

where h is the departure of formation's hydraulic head from static conditions, K_r the hydraulic conductivity in the radial direction, S_s the specific storage, b the formation thickness, H the departure of the head in the well from static conditions, H_o the head departure in the well after instantaneous initial displacement, r_s the effective radius of well screen, r_c the effective radius of well casing, r the radial distance from the well axis, t the time, n the porosity, and ρ the mass density of water.

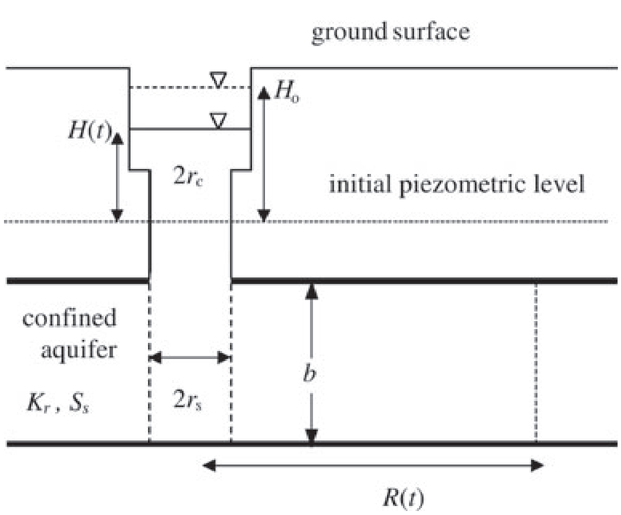

Figure 1. Slug test in a well fully penetrating a confined aquifer.

Initial and boundary conditions for the well and for the aquifer are also needed. The initial conditions imply an instantaneous introduction of the slug and are as follows:

$$h(r, 0) = 0, r_s < r < \infty \text{ (in the aquifer);} \tag{2a}$$

$$H(0) = H_o \text{ (in the well).} \tag{2b}$$

It is furthermore assumed that no well losses occur (no *skin effect*), whence at the well screen

$$h(r_s, t) = H(t), t > 0. \tag{3a}$$

In addition, the flow across the well screen (into/out of the aquifer) equals the rate of volume change of the water column in the well, where hydrostatic conditions are assumed to prevail:

$$2\pi r_s K_r b \frac{\partial h(r,t)}{\partial r}\bigg|_{r_s} = \pi r_c^2 \frac{\mathrm{d}H(t)}{\mathrm{d}t}. \tag{3b}$$

Finally, the aquifer's boundaries are taken at an infinite radial distance from the test well:

$$h(r \to \infty, t) = 0, t > 0. \tag{4}$$

Equations 1 through 4 comprise the mathematical model of the method of Cooper et al. (1967). Its analytical solution (integral of Bessel functions) is written in terms of the dimensionless parameters for time, β, and for storage, α, and has been tabulated by Cooper et al. (1967) and Papadopulos et al. (1973) in the range of practical interest:

$$\frac{H(t)}{H_o} = f(\beta, \alpha), \tag{5}$$

$$\beta = \frac{K_r b t}{r_c^2}, \tag{6}$$

$$\alpha = \frac{r_s^2 S_s b}{r_c^2}. \tag{7}$$

The Quasi-Steady Flow Model of Hvorslev (1951)

Hvorslev simplified the problem by assuming that the specific storage is so small that the right-hand side of the governing Equation 1 may be set to zero without great loss in accuracy,

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = 0. \tag{8}$$

The governing equation has thus become quasi-steady; this approximation causes any change at the well to propagate instantly throughout the aquifer. One implication is that the flow rate through cylindrical surfaces of any radius r around the well axis is the same. Another result is that the constant-head ($h = 0$) boundary may not be placed at infinity, but at an unknown and time-varying distance R instead; accordingly, for $t > 0$, Equation 4 is replaced by

$$h(r = R) = 0. \tag{9}$$

Of course, transient flow cannot be represented by a quasi-steady flow equation, in which time does not appear explicitly. But this obstacle is circumvented by writing Equation 3b, which expresses the aforementioned constancy of the flow rate, for an arbitrary radial distance r; that equation is also restated by introducing $r\partial h/\partial r = \partial h/\partial \ln r$:

$$2\pi r K_r b \frac{\partial h(r,t)}{\partial r} = 2\pi K_r b \frac{\partial h(r,t)}{\partial \ln r} = \pi r_c^2 \frac{\mathrm{d}H(t)}{\mathrm{d}t} \tag{10a}$$

$$\frac{\partial h(r,t)}{\partial \ln r} = \left(\frac{r_c^2}{2K_r b}\right)\frac{\mathrm{d}H(t)}{\mathrm{d}t}. \tag{10b}$$

Equations 3a and 8 through 10 comprise the mathematical model of the method of Hvorslev (1951), which differs from that of Cooper et al. (1967) in that (Butler 1998): (1) it ignores the effect of storage, (2) the constant-head boundaries are placed at the constant distance R, and (3) the slug must no longer be introduced instantly. The solution of Hvorslev's model is derived as follows. Because the right-hand side of Equation 10b is independent of r, so is $\partial h/\partial \ln r$; it then holds exactly

$$\frac{\partial h}{\partial \ln r} = \frac{\Delta h}{\Delta \ln r} = \frac{h(R) - h(r_s)}{\ln R - \ln r_s} = \frac{0 - H(t)}{\ln(R/r_s)}. \tag{11}$$

We may now express the left-hand side of Equation 10b with Equation 11 to obtain

$$-\frac{H(t)}{\ln(R/r_s)} = \frac{r_c^2}{2K_r b}\frac{dH}{dt}, \tag{12}$$

which is integrated analytically, yielding the equation of Hvorslev

$$\ln\left[\frac{H(t)}{H_o}\right] = -\frac{2K_r bt}{r_c^2 \ln(R/r_s)}. \tag{13}$$

This quasi-steady model solution plots as a straight line on a logarithmic-linear graph of the normalized head recovery at the well vs. time, and the value of K_r is obtained from the slope of the best straight-line fit of the data through regression. A common approach is to estimate this slope from the *basic time lag* T_o at which $\ln(H/H_o) = 0.368$, exploiting the identity $\ln(0.368) = -1$; since, at $t = 0$, $\ln(H/H_o) = 0$ and $2K_r bt/[r_c^2 \ \ln(R/r_s)] = 0$, the slope is $\ln(0.368)/T_o = -1/T_o$. Finally, solving Equation 13 for K_r, with $\ln(H/H_o) = -1$ and $t = T_o$, gives

$$K_r = \frac{r_c^2 \ln(R/r_s)}{2bT_o}. \tag{14}$$

Butler (1998) shows that estimation accuracy improves when a straight line is fitted through the nearly linear mid-portion of the data set ($0.15 \le H/H_o \le 0.25$) and the data are renormalized as $H(t)/H_o/H_o^+$, where H_o^+ is that straight line's intercept on the axis of $\log[H(t)/H_o]$.

Equation 14 can be evaluated, if the value of $\ln(R/r_s)$ is known. Chirlin (1989) showed that: (1) while the optimal value of $\ln(R/r_s)$ is unknown, the error in the estimation of K_r is within a factor of 10, when the elastic storage parameter α varies over 10 orders of magnitude and $4 \le R/r_s \le 320 \times 10^3$; and (2) the omission of storage in Hvorslev's method leads to greater sensitivity of the results the more compressible the aquifer is; an upward-concave curve of $\log(H/H_o)$ against t indicates that storage plays a role and care must be exercised in fitting a straight line through the response data. Ostendorfer and DeGroot (2010) show that curved slug test responses can also result due to a slowly (seasonally) varying background head occasionally present in leaky or consolidating aquifers, and typically in unconfined aquifers of very low permeability.

The Complete Quasi-Steady Flow Method

The quasi-steady flow Equation 8, with boundary conditions Equation 3a (with $H = h(r_s)$, at the $h(r_s)$-value corresponding to each time t) and Equation 9, is integrated, yielding the analytical solution

$$\frac{h(r)}{h(r_s)} = \frac{h(r)}{H} = \frac{\ln(r/R)}{\ln(r_s/R)} = \frac{\ln(R/r)}{\ln(R/r_s)}. \tag{15}$$

We estimate the specific storage coefficient S_s by applying the principle of mass conservation, using the quasi-steady solution Equation 15 as follows. Under the assumption of quasi-steady flow, at any time t after the initiation of the slug test, the change of the water volume stored inside the well casing relative to the initial slug volume, V_{well}, equals the change in water stored in the aquifer up to $r = R(t)$ relative to the initial aquifer volume, V_{aq}, since that slug water entered the aquifer flowing across the well screen. The former water volume is simply

$$V_{well}(t) = \pi r_c^2[H(t) - H_o]. \tag{16}$$

The latter is calculated by integrating the quasi-steady flow profile, Equation 15, over the volume bounded by the cylindrical aquifer surfaces at $r = r_s$ and $r = R(t)$, multiplied by S_s, or

$$V_{aq} = 2\pi b S_s \int_{r_s}^{R} h(r) r \mathrm{d}r$$
$$= \frac{2\pi b S_s R^2}{\ln(R/r_s)} \int_1^{r_s/R} \ln(r/R)(r/R)\mathrm{d}(r/R). \qquad (17)$$

In Equation 17, with $x = r/R$, $\int x \ln x \, \mathrm{d}x = x^2[\ln(x)/2 - 1/4]$; therefore the expression for V_{aq} becomes

$$V_{aq} = \frac{2\pi b S_s H R^2}{\ln(R/r_s)} \left(\frac{1}{4} - \frac{(r_s/R)^2}{4}[2\ln(R/r_s) + 1]\right)$$
$$= \pi b S_s H R^2 \left(\frac{(R(t)/r_s)^2 - 2\ln(R(t)/r_s) - 1}{2\ln(R(t)/r_s)}\right). \qquad (17a)$$

Mass conservation, under the quasi-steady flow assumption, $V_{well} = V_{aq}$, yields then

$$\frac{H_o - H(t)}{H(t)} = \frac{b S_s r_s^2}{r_c^2} \left(\frac{(R(t)/r_s)^2 - 2\ln(R(t)/r_s) - 1}{2\ln(R(t)/r_s)}\right), \qquad (18)$$

or restated using the dimensionless storage parameter $\alpha = r_s^2 S_s b / r_c^2$ of Cooper et al. (1967):

$$\frac{H_o - H(t)}{H(t)} = \alpha \left(\frac{(R(t)/r_s)^2 - 2\ln(R(t)/r_s) - 1}{2\ln(R(t)/r_s)}\right). \qquad (19)$$

Finally, the statement of mass conservation is supplemented by Equation 12, which expresses the constancy of flow rate and is rearranged slightly as follows,

$$\frac{1}{H(t)} \frac{\mathrm{d}H(t)}{\mathrm{d}t} = \frac{\mathrm{d}\ln[H(t)/H_o]}{\mathrm{d}t} = -\frac{2K_r b}{r_c^2 \ln[R(t)/r_s]}, \qquad (20)$$

or restated in terms of the dimensionless time parameter $\beta = K_r b t / r_c^2$ of Cooper et al. (1967),

$$\frac{\mathrm{d}\ln[H(\beta)/H_o]}{\mathrm{d}\beta} = -\frac{2}{\ln[R(\beta)/r_s]}. \qquad (21)$$

Note that in Equation 20 $R = R(t)$, while, in the otherwise identical solution of the Hvorslev model, Equation 13, R is assumed to be constant. Chirlin (1989) gives the following interpretation of the slug test's effective radius necessitated by the incompressible approximation in Hvorslev's model: "This approximation spawns an adjustable parameter r_e in Hvorslev (1951) that provides a degree of freedom analogous to the physically based compressive storage of Cooper et al. (1967)" (r_e corresponds to R here). We emphasize that the complete quasi-steady flow model not only remedies the deficiency of the Hvorslev model, by permitting the estimation of the specific storage, but also eliminates the empirical concept of the slug test's *constant* effective radius (Butler 1998) considered by Chirlin (1989) as a substitute storage parameter.

Solving Equations 18 and 20 (or Equations 19 and 21) numerically (see Appendix), we generate type-curves, with α as parameter, depicting the departure from static conditions of the water level in the well normalized by the initial displacement, H/H_o, as function of β in logarithmic scale. The complete quasi-steady model is compared to the Cooper et al. (1967) standard in Figure 2. The good agreement testifies to its accuracy, supporting Hvorslev's contention of near-incompressible flow dynamics. The small discrepancies below $H/H_o \approx 0.1$ are due to the two models' different constant-head boundaries, $R(t)$ and $r \to \infty$. In the quasi-steady approximation, the flow entering the aquifer at the well screen propagates instantly to $r = R(t)$, where it exits; the boundary $R(t)$ responds to the flow at the well screen. The fundamental premise of the complete quasi-steady flow model, $V_{well} = V_{aq}$, is inexact because Equation 8 and its solution Equation 15 ignore elastic storage, the results however show this effect to be small and adequately captured by the variable R.

The complete quasi-steady flow model is not simply a second step for calculating the specific storage, after estimating the radial hydraulic conductivity by Hvorslev's model. It is a self-contained method for simultaneously estimating a formation's hydraulic parameters K_r and S_s, in which the radius of influence R varies temporally, as Equations 18 through 21 reflect. Importantly, quasi-steady flow type-curves for specified values of α are generated easily using an electronic spreadsheet. One would then apply the matching procedure of Cooper et al. (1967) to calculate the formation's hydraulic parameters: (1) fit the observations optimally with a type-curve (α_{opt}) (abscissas must have scales with equal number of log cycles), (2) select a convenient *match point*—for example, $\beta = 1$, for which the time $t_{\beta=1.0}$ is read off the x-axis of the data

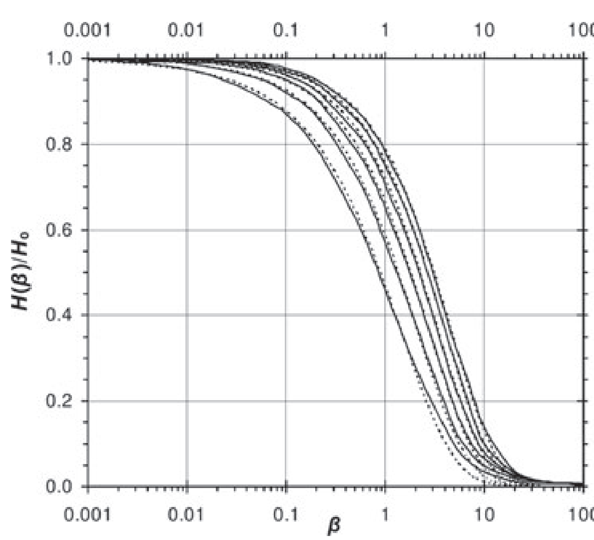

Figure 2. Water level in the well as function of the dimensionless time parameter β, for values of the dimensionless storage parameter $\alpha = 10^{-2}$, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, and 10^{-7}, decreasing upwards: solid lines, complete quasi-steady flow model; dashed lines, model of Cooper et al. (1967).

plot—and (3) use Equations 6 and 7 with $\beta = 1$, $t_{\beta=1.0}$, and α_{opt} to compute $K_r = r_c^2/bt_{\beta=1.0}$ and $S_s = \alpha_{opt} r_c^2/r_s^2 b$.

Field Verification

We use the data from a slug test performed in well Ln1 at the Lincoln County, Kansas monitoring site—Table 3.1 in Butler (1998)—to demonstrate the application of the complete quasi-steady flow model. The well penetrates fully the confined aquifer of thickness $b = 3.05$ m and has effective screen radius $r_s = 0.071$ m and effective radius of casing $r_c = 0.025$ m. The model of Cooper et al. (1967) provides an excellent fit to those data for $\alpha_{opt} = 0.0108$—Figure 5.4 in Butler (1998)—yielding $S_s = 4.38 \times 10^{-4}$/m, and, from the match point $\beta = 1$, $K_r = 3.69 \times 10^{-4}$ m/d. These results are the reference standard. For comparison, we also note two values of the radial hydraulic conductivity estimated by Hvorslev's method with $R/r_s = 200$, as recommended by the U.S. Navy (Butler 1998): (1) $K_r = 4.65 \times 10^{-4}$ m/d, when the linear regression honors all data points, and (2) $K_r = 3.88 \times 10^{-4}$ m/d, by fitting a straight line only through the nearly linear mid-portion of the data and renormalizing the data (Butler 1998).

Figure 3 shows the test data fit by the quasi-steady model's type-curve with $\alpha_{opt} = 0.0111$ (visually) that yields the specific storage estimate $S_s = 4.51 \times 10^{-4}$/m via Equation 7. From the match point at $\beta = 1$, $t_{\beta=1.0} = 47863$ s, we estimate the radial hydraulic conductivity as $K_r = 3.70 \times 10^{-4}$ m/d. Both parameter values are very close to the reference standard of Cooper et al. (1967).

We have also tested an optimization method that minimizes the bias (mean deviation) of the computed curve from the data points, while conditioning the first to have the same area as the data curve, both plotted in logarithmic time scale; these indicators are theoretically identical, but their values differ in applications, as they depend on the distribution and number of data points. The quasi-steady flow solution is computed repeatedly in an *exhaustive-search* (Mills 2010), testing α- and β-values until finding the optimal pair. The simplicity of the quasi-steady flow solution allowed executing the optimization on an electronic spreadsheet. The obtained formation parameter values $K_r = 3.67 \times 10^{-4}$ m/d and $S_s = 5.10 \times 10^{-4}$/m are close to the visually optimized parameters of the quasi-steady flow and the Cooper et al. (1967) methods.

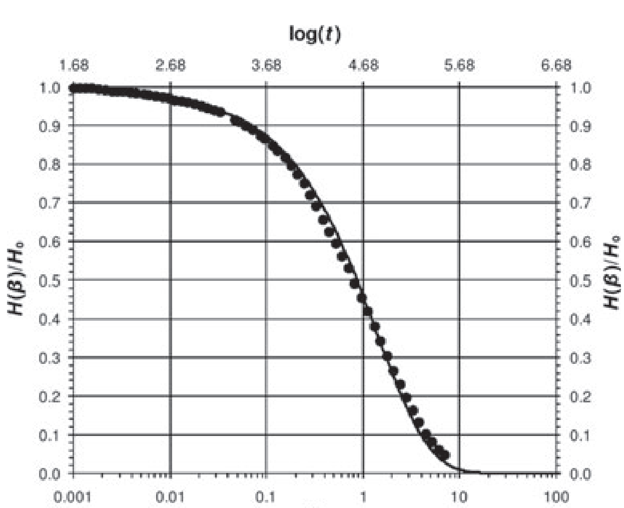

Figure 3. Fit of test data (•••) by the complete quasi-steady model's type-curve with $\alpha_{opt} = 0.0111$.

Conclusions

The complete quasi-steady flow model, with a time-variable radius of influence R, underpins an efficient method for estimating both hydraulic parameters K_r and S_s of a formation in the over-damped case. This model extends and completes the model of Hvorslev (1951), for it uses the same quasi-steady flow approximation, yet enables also estimating a formation's specific storage coefficient via a transient aquifer storage balance. The accuracy of the quasi-steady flow model has been tested and verified against the model standard of Cooper et al. (1967), using field data from a slug test performed at a monitoring site of the Kansas Geological Survey. Type-curves of this quasi-steady flow model are generated easily using an electronic spreadsheet. The advantage of this computational ease and efficiency is that the optimal fit of data by a type-curve can be readily automated for practical applications.

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Appendix

The Calculation of the Quasi-Steady Flow Model's Type-Curves $H/H_o = f(\alpha, \beta)$

Values of the ratio $y = R/r_s$ are computed in increments Δy from 1 to, say, 10^6 (Δy is not necessarily constant and may be increased as y increases) and the right-hand side of Equation 19

$$F(y) = \frac{(y^2 - 2\ln y - 1)}{2\ln y} \tag{A1}$$

is calculated for each y; application of the rule of de L' Hospital verifies that $F(y = 1) = 0$. According to the Equation 19, for any value of the storage parameter α the value of the expression $[\alpha F(y) + 1]^{-1}$ represents a specific instance of H/H_o at time t (or $\beta = K_r bt/r_c^2$) through

$$\frac{H(t)}{H_o} = \frac{H(\beta)}{H_o} = [\alpha F(y) + 1]^{-1}. \tag{A2}$$

$H(\beta)/H_o$ is linked to appropriate β-values by calculating via Equation 21 for each $y = y_i$ the associated

$$\Delta\beta_{i/i+1} = \frac{-\ln y_i \cdot \Delta[\ln(H(\beta_i)/H_o)]}{2} = \frac{-\ln y_i \cdot \Delta \ln[\alpha F(y_i) + 1]^{-1}}{2}; \quad \text{(A3)}$$

$\Delta\beta_{i/i+1}$ is the interval between time points i and $i + 1$. The β-sequence associated with the y-values is generated starting at $y = 1$, which corresponds to $\beta = 0$. This procedure may be readily carried out with an electronic spreadsheet by: (1) storing the values of r_s, r_c, b, K_r, and S_s in cells B1..., B5 and computing α in cell B6, say; and (2) calculating row-by-row, for example, in columns D-H, y in D and $F(y)$ in E, β in F and t in G, and H/H_o in H. This simple solution by forward finite differences (Euler scheme) typically requires small steps for accuracy; if desired, however, other self-starting methods, such as Runge-Kutta schemes, may be used instead.

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