Boundary-Layer Meteorology (2005) 115: 381–398
DOI 10.1007/s10546-004-1426-3

INTERPOLATION BETWEEN BUSINGER–DYER FORMULAE AND FREE CONVECTION FORMS: A REVISED APPROACH

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(Received in final form 14 August 2004)

Abstract. In this study, profile functions for flux calculations during unstable conditions are proposed and examined. These functions are based on a direct interpolation for the dimensionless wind speed and temperature gradients between the standard Businger–Dyer formulae, $\phi_{\mathrm{K}u}(\zeta) = (1-\gamma_u\zeta)^{-1/4}$, $\phi_{\mathrm{K}t}(\zeta) = (1-\gamma_t\zeta)^{-1/2}$, and free convection forms, $\phi_{\mathrm{C}u,t}(\zeta) = (1-\alpha_{\mathrm{C}u,t}\zeta)^{-1/3}$, ζ being the Monin–Obukhov stability parameter. A previously presented interpolation between the corresponding profile relationships, in attempting to provide a general relationship for the whole unstable regime, leads to serious restrictions for the values of $\alpha_{\mathrm{C}u,t}$ in the free convection forms. These restrictions rendered available experimental data almost inapplicable, since the behaviour of the formulae in the near-neutral range controls the values of those parameters. The proposed interpolation provides functions that, firstly, fit the standard Businger–Dyer forms for near-neutral conditions and, secondly, satisfy the asymptotic behaviour as $\zeta \to -\infty$, permitting wider ranges of possible $\alpha_{\mathrm{Cu},t}$ values. This step is very important, taking into account the large spread of the experimental data. Thus, as further and more accurate observations at strong instability become available, this approach could prove very efficient in fitting these data while retaining correct near-neutral behaviour.

Keywords: Businger–Dyer formulae, Convective forms, Free convection, Monin–Obukhov theory.

1. Introduction

The Monin–Obukhov (M–O) similarity theory (Obukhov, 1946; Monin and Obukhov, 1954) is the most widely accepted way to describe turbulence and vertical turbulent fluxes in the horizontally homogeneous and stationary atmospheric surface layer. According to this theory, the dimensionless vertical gradients for mean wind speed and temperature are universal functions of the dimensionless stability parameter,

$$\zeta = \frac{z}{L}, \tag{1}$$

where $z\,(\ll h)$ is the height above the surface, h is the boundary-layer height and L is the Obukhov length, which is given by,

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$$L = \frac{-u_*^3 T_v}{kg\overline{w'\theta'_v}}. \tag{2}$$

Here u_* is the friction velocity, T_v is the virtual temperature, k is the von Kármán constant, g is the gravitational acceleration and $\overline{w'\theta'_v}$ the virtual temperature flux at the surface. The dimensionless gradients of wind speed and temperature are determined through the following equations:

$$\phi_u(\zeta) = \left(\frac{kz}{u_*}\right)\frac{d\overline{U}}{dz}, \tag{3a}$$

$$\phi_t(\zeta) = \left(\frac{kz}{\theta_*}\right)\frac{d\overline{\theta}}{dz}, \tag{3b}$$

where θ_* is the temperature scale given by

$$\theta_* = \frac{-\overline{w'\theta'}}{u_*}. \tag{4}$$

The forms $\phi_{u,t}(\zeta)$ are universal functions of the dimensionless stability parameter ζ. Their general mathematical form cannot be predicted by the M–O theory, so the functions are estimated through experimental measurements. However, the theory provides some constraints on the asymptotic forms of $\phi_{u,t}(\zeta)$. Thus, during neutral stability conditions ($\zeta \to 0$), both functions approach unity, corresponding to the well-known logarithmic profile case. In the free convection limit ($-\zeta \to \infty$), it has been suggested (Obukhov, 1946, 1959; Monin and Obukhov, 1954; Priestley, 1954, 1955; Monin, 1959) that the surface stress and, correspondingly u_* can be dropped from the list of independent variables. Dimensional analysis shows that in this case the dimensionless gradients approach the following convective dependence:

$$\phi_{u,t}(\zeta) = A_{u,t}(-\zeta)^{-1/3}. \tag{5}$$

It is noted that the convective parameters, $A_{u,t}$, are universal constants and have to be determined through measurements. However, measurements in free convection are difficult to carry out and most determinations are based on observations that correspond to values of ζ close to the neutral regime. Such a limited area of instability also has been used to date for the estimation and evaluation of the widely used Businger–Dyer (Kansas-type) functions (Businger, 1966; Dyer and Hicks, 1970; Businger et al., 1971; Dyer, 1974),

$$\phi_{Ku}(\zeta) = (1 - \gamma_u\zeta)^{-1/4}, \tag{6a}$$

$$\phi_{Kt}(\zeta) = (1 - \gamma_t\zeta)^{-1/2}, \tag{6b}$$

which are based on measurements from the famous Kansas experiment (Izumi, 1971; Izumi and Coughey, 1976). These forms satisfy the neutral limit but do not show the predicted asymptotic behaviour of (5) for large negative values of ζ. Their validity lies in the range $-2 < \zeta < 0$, although some researchers argue that this range could be extended to -5 (Garratt, 1992) or even to -10 (Dyer and Bradley, 1982). The proper choices for $\gamma_{u,t}$ have been the subject of extended discussion (Yaglom, 1977; Dyer and Bradley, 1982; Hogstrom, 1985, 1986, 1988; Telford and Businger, 1986; Businger, 1988; Sorbjan, 1989; Garratt, 1992), with the values $\gamma_u = \gamma_t = 16$ by far the most commonly used.

An alternative convective form that has been suggested (Carl et al., 1973; Fairall et al., 1996) results from the replacement of the exponents in (6) by $-1/3$:

$$\phi_{Cu,t}(\zeta) = (1 - \alpha_{Cu,t}\zeta)^{-1/3}. \tag{7}$$

Such forms coincide with the asymptotic character of (5) for $\alpha_{Cu} = A_u^{-3}$ and $\alpha_{Ct} = A_t^{-3}$, but their behaviour at near-neutral stability is not necessarily correct, since the theory supports the value $-1/3$ for the exponents only during strong instability. In addition, there is a lack of experimental data in the free convection regime. As a result, numerous choices for the values of $\alpha_{Cu,t}$ have been suggested (see Grachev et al., 2000 for an overview), after evaluation, mainly, of observations made under moderate instability conditions.

2. Wind Speed and Temperature Flux–Profile Relations

The vertical profiles of wind speed and temperature could be calculated through the integration of Equations (3) (Panofsky, 1963) from a height equal to the roughness length for wind, z_{0u}, or for temperature, z_{0t}, (Brutsaert, 1975; Garratt et al., 1993) to the generalised height of application, z. However, the following integration of the $\phi(\zeta)$ functions is mostly used instead:

$$\Psi_{u,t}(\zeta) = \int_0^{\zeta} \frac{1 - \phi_{u,t}(\xi)}{\xi} d\xi. \tag{8}$$

The use of this integral makes it possible to write the vertical profiles as deviations from the neutral logarithmic profile in the form

$$\begin{aligned} \overline{U}(z) &= \frac{u_*}{k}\left[\ln\left(\frac{z}{z_{0u}}\right) - \Psi_u(\zeta) + \Psi_u(\zeta_{0u})\right], \\ \overline{\theta}(z) - \theta_0 &= \frac{\theta_*}{k}\left[\ln\left(\frac{z}{z_{0t}}\right) - \Psi_t(\zeta) + \Psi_t(\zeta_{0t})\right], \end{aligned} \tag{9}$$

where ζ_{0u} and ζ_{0t} are the values of the stability parameter, $\zeta = z/L$, at the heights z_{0u} and z_{0t} respectively. Here, it must be pointed out that this study is not concerned with the possible difference between the values for k used for wind and temperature profiles (Zilitinkevich et al., 1998), since this does not affect the general concept.

If Equation (8) is applied to the Kansas-type $\phi_K(\zeta)$ forms (6), the corresponding $\Psi_{Ku}(\zeta)$ for the wind-speed profile function becomes (Panofsky, 1963; Paulson, 1970)

$$\Psi_{Ku}(\zeta) = 2\ln\left(\frac{1+x}{2}\right) + \ln\left(\frac{1+x^2}{2}\right) - 2\arctan\, x + \frac{\pi}{2}, \tag{10}$$

where $x = (1 - \gamma_u \zeta)^{1/4}$. The corresponding calculation of $\Psi_{Kt}(\zeta)$ yields

$$\Psi_{Kt}(\zeta) = 2\ln\left(\frac{1+\sqrt{1-\gamma_t \zeta}}{2}\right). \tag{11}$$

In the case of the alternative convective forms, given by (7), Fairall et al. (1996) calculated the respective expressions via (8) as

$$\Psi_{Cu,t}(\zeta) = \frac{3}{2}\ln\left(\frac{y^2+y+1}{3}\right) - \sqrt{3}\arctan\left(\frac{2y+1}{\sqrt{3}}\right) + \frac{\pi}{\sqrt{3}}, \tag{12}$$

where $y = (1 - \alpha_{u,t}\zeta)^{1/3}$.

Unlike the aforementioned functions (6) and (7), the use of the fully convective forms (5) could not lead to Ψ expressions, since they are not valid for ζ values close to 0. If the integration, from $z_{0u,t}$ to z, is applied directly to (5) instead, the resulting profiles show a dependence close to $z^{-1/3}$, which extends down to the surface:

$$\overline{U}(z) = \frac{3A_u u_*}{k}\left((-\zeta_{0u})^{-1/3} - (-\zeta)^{-1/3}\right), \tag{13a}$$

$$\overline{\theta}(z) - \theta_0 = \frac{3A_t \theta_*}{k}\left((-\zeta_{0t})^{-1/3} - (-\zeta)^{-1/3}\right). \tag{13b}$$

Equations (13), for the vertical profiles, do not have a logarithmic part. This holds true also in the case of the profiles given by the combination of Equations (9) and (12), but only for large values of ζ_0 (Grachev et al., 1997, 1998; Akylas et al., 2001). For lower values of ζ, however, the use of such functions is debatable, since, in that case, the behaviour of (7) is not necessarily appropriate.

3. Interpolating between Kansas-Type and Convective Formulae

Among other techniques (e.g., Wilson, 2001), a reasonable way to produce functions that cover the whole stability range, from neutral to free convection, is to interpolate between the Kansas-type (K) and convective (C) formulae (Fairall et al., 1996, 2003; Grachev et al., 2000). The resulting equations intend to show a good overall behaviour, by overlapping the Kansas-type functions in the near-neutral range and, at the same time, satisfying the correct asymptotic behaviour.

3.1. Interpolation between flux–profile relations

Fairall et al. (1996, 2003) have chosen to interpolate between the flux–profile relations (10), (11) and (12), through the following formulation:

$$\Psi_{1u,t}(\zeta) = \frac{\Psi_{Ku,t}(\zeta) + \zeta^2 \Psi_{Cu,t}(\zeta)}{1+\zeta^2}. \tag{14}$$

In order to calculate the dimensionless gradients $\phi(\zeta)$, corresponding to a known flux–profile relation $\Psi(\zeta)$, Equation (8) must be transformed to

$$\phi(\zeta) = 1 - \zeta \frac{d\Psi(\zeta)}{d\zeta}. \tag{15}$$

By introducing $\Psi_{1u,t}(\zeta)$ from (14) into (15), the corresponding dimensionless gradients, $\phi_{1u,t}(\zeta)$, become:

$$\phi_{1u,t}(\zeta) = \frac{\phi_{Ku,t}(\zeta) + \zeta^2 \phi_{Cu,t}(\zeta)}{1+\zeta^2} + 2\zeta^2 \frac{\left(\Psi_{Ku,t}(\zeta) - \Psi_{Cu,t}(\zeta)\right)}{\left(1+\zeta^2\right)^2}. \tag{16}$$

From the above expression, it is clear that ϕ_1 is very dependent on the difference between Ψ_K and Ψ_C, which may cause ϕ_1 to deviate from a smooth overall shape. Grachev et al. (2000) examined the first derivative of (16) and showed (in their Figures 3 and 4) that this interpolation leads to smooth and monotonically decreasing functions for only the specific choice of $\alpha_{Cu} = 10$ and $\alpha_{Ct} = 34$. As will be shown here, the above limitation is a result of the strong dependence of the $\Psi_{Cu,t}(\zeta)$ values, from (12), on the shape and the magnitude of the corresponding $\phi_{Cu,t}(\zeta)$ function (7) close to the neutral limit; this is true even for strong convection. The choice of the flux–profile relations Ψ for the interpolation (14) includes information about the behaviour of the dimensionless gradients ϕ_C during near-neutral conditions. In order for the resulting function to be smooth and monotonic, $\Psi_{Cu,t}(\zeta)$ are forced to approach the respective Kansas-types for small values of the parameter $-\zeta$. As a result, the values for the parameters $\alpha_{Cu,t}$ are evaluated indirectly, over data representative of light instabilities. However, the behaviour of the functions Ψ and ϕ for low instability is known to coincide with the Kansas-type formulae given by (6), (10) and (11). Thus, any interpolation attempted should aim to keep the lower part of the resulting functions as close as possible to the Kansas estimates.

3.2. Interpolating between dimensionless gradients

In this study, an alternative, direct interpolation between $\phi_{\mathrm{K}}(\zeta)$ and $\phi_{\mathrm{C}}(\zeta)$ is investigated:

$$\phi_{u,t}(\zeta) = \frac{c^2\phi_{\mathrm{K}u,t}(\zeta) + \zeta^2\phi_{\mathrm{C}u,t}(\zeta)}{c^2+\zeta^2}. \tag{17}$$

The main idea is that such an interpolation diminishes the influence of the shape of the convective part of the function in the near-neutral stability area. The near-neutral part of the resulting function is kept close to the Kansas results, thereafter continuing with the proper asymptotic behaviour towards the free convection limit. Parameter c corresponds to a critical value of ζ above which the influence of the convective formulae, ϕ_{C}, starts to become more important than that of the Kansas-type form, ϕ_{K}. The value of $c = 1$ was used in all calculations of this study. Different choices of that parameter could be used, in order to obtain a better performance and a smoother behaviour of the resulting function, wherever it is necessary.

In order to ensure the smooth and monotonic behaviour of (17) over ζ, its first derivative,

$$\frac{d\phi_{u,t}(\zeta)}{d\zeta} = \frac{c^2\phi'_{\mathrm{K}u,t}(\zeta) + \zeta^2\phi'_{\mathrm{C}u,t}(\zeta)}{c^2+\zeta^2} - \frac{2c^2\zeta\left(\phi_{\mathrm{K}u,t}(\zeta) - \phi_{\mathrm{C}u,t}(\zeta)\right)}{\left(c^2+\zeta^2\right)^2}, \tag{18}$$

has been calculated and tested to be positive and monotonic; in (18), $\phi'_{\mathrm{K}u,t}(\zeta)$ and $\phi'_{\mathrm{C}u,t}(\zeta)$ are the first derivatives of (6) and (7) (or (5)), respectively.

In Figure 1, the forms of $\phi(\zeta)$ and $d\phi(\zeta)/d\zeta$ from the interpolation between Equations (6) and (7), for both wind speed and temperature, are illustrated, for a variety of $\alpha_{\mathrm{C}u,t}$ choices. For the chosen values of the convective constants $\alpha_{\mathrm{C}u} > 5$ and $\alpha_{\mathrm{C}t} > 15$, the resulting functions $\phi_{u,t}(\zeta)$ exhibit a smooth and physically acceptable shape. This holds also true when, instead of (7), (5) with $A_u = \alpha_{\mathrm{C}u}^{-1/3}$ and $A_t = \alpha_{\mathrm{C}t}^{-1/3}$, is used in the interpolation (not shown here). In fact, the resulting functions differ from each other by less than 1% for values of the convective constants greater than 5. As a result, it does not matter which convective form is used, but the specific choice for the values of $\alpha_{\mathrm{C}u,t}$ is important.

It must be noted that the mathematical demand for continuous, monotonic derivatives of (17) narrows the range of the possible values for the convective constants, as the order of the derivative increases. This is more profound in the case of the upper limit of those constants, although the use of larger values of parameter c may soften the restrictions. If, however, all the derivatives, of any order, would be calculated (this has not been considered necessary), the above demand would have led to a unique set of $\alpha_{\mathrm{C}u,t}$ values (as in the case of Grachev et al., 2000). Nevertheless, this procedure does not solve the principal problem that there will always be a non-monotonic

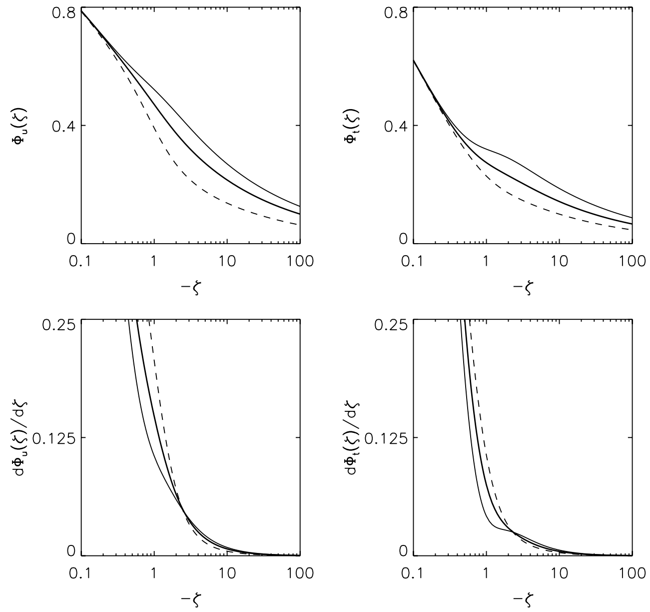

Figure 1. $\phi_{u,t}(\zeta)$ given by interpolating through (17) between (6) and (7) and their first derivatives, $d\phi_{u,t}(\zeta)/d\zeta$ given by (18), for different choices of parameter $\alpha_{Cu} = 5$ (continuous thin lines), 10 (continuous heavy lines), 40 (dashed lines) and $\alpha_{Ct} = 15$ (continuous thin lines), 34 (continuous heavy lines), 100 (dashed lines). In all cases $\gamma_{u,t} = 16$ and $c = 1$.

derivative of higher order, which is a result of the interpolation between functions with different exponents. Thus, from one point of view, there is no physical difference between a choice of the convective constants, that satisfies the $(n-1)$th derivative and another that satisfies the nth, if the $(n+1)$th derivative is not satisfied at all. In this study, it has been considered sufficient that the interpolation (17) leads clearly to much smoother functions compared to (16), which is affected by the near-neutral behaviour of (7).

4. Analytical Relations for $\Psi_{u,t}$ Functions

A problem that arises from the use (17) is the calculation of the corresponding flux–profile relations $\Psi_{u,t}(\zeta)$. By introducing the gradients given by (17) into (8), the general solution can be written as

$$\Psi_{u,t}(\zeta) = A_{u,t}(\zeta) - A_{u,t}(0) + B_{u,t}(\zeta) - B_{u,t}(0), \tag{19}$$

where $A_{u,t}(\zeta)$ and $B_{u,t}(\zeta)$ are determined from

$$A_{u,t}(\zeta) = \int \frac{d\zeta}{\zeta} - \int \frac{c^2 \phi_{Ku,t}(\zeta)}{\zeta\left(c^2 + \zeta^2\right)} d\zeta, \tag{20}$$

which corresponds to the influence from the Kansas forms (6), and from

$$B_{u,t}(\zeta) = -\int \frac{\zeta\phi_{\mathrm{C}u,t}(\zeta)}{\left(c^2+\zeta^2\right)} d\zeta, \tag{21}$$

the part of the solution inserted due to the convective forms (5) or (7).

The above integrals can be easily calculated through numerical integration. However, in order to assure accuracy, an analytical solution has been derived herein, with $\alpha_{\mathrm{C}u,t}$, $\gamma_{u,t}$ and c as independent, free variables. More specifically, in the case of the wind-speed profile, the integral $A_u(\zeta)$ is given from the following formulation:

$$\begin{aligned} A_u(\zeta) = {} & \left(\frac{1+\gamma_{u1}\left(\gamma_{u3}^2-\gamma_{u1}\right)}{2\gamma_{u1}\gamma_{u2}\gamma_{u3}}\right)\ln\left(\frac{x^2+\gamma_{u3}x+\gamma_{u1}}{x^2-\gamma_{u3}x+\gamma_{u1}}\right)+\left(\frac{1+\gamma_{u1}\left(\gamma_{u4}^2-\gamma_{u1}\right)}{2\gamma_{u1}\gamma_{u2}\gamma_{u4}}\right) \\ & \times\ln\left(\frac{x^2-\gamma_{u4}x+\gamma_{u1}}{x^2+\gamma_{u4}x+\gamma_{u1}}\right)+\left(\frac{3\gamma_{u1}^2-\gamma_{u1}\gamma_{u3}^2+1}{\gamma_{u1}\gamma_{u2}\gamma_{u4}}\right)\left[\arctan\left(\frac{2x+\gamma_{u3}}{\gamma_{u4}}\right)\right. \\ & \left.+\arctan\left(\frac{2x-\gamma_{u3}}{\gamma_{u4}}\right)\right]-\left(\frac{3\gamma_{u1}^2-\gamma_{u1}\gamma_{u4}^2+1}{\gamma_{u1}\gamma_{u2}\gamma_{u3}}\right)\left[\arctan\left(\frac{2x+\gamma_{u4}}{\gamma_{u3}}\right)\right. \\ & \left.+\arctan\left(\frac{2x-\gamma_{u4}}{\gamma_{u3}}\right)\right]+\ln\left(x^2+1\right)+2\ln(x+1)-2\arctan(x), \end{aligned} \tag{22}$$

where the introduced variables are $x=(1-\gamma_u\zeta)^{1/4}$, $\gamma_{u1}=\left(1+\gamma_u^2c^2\right)^{1/4}$, $\gamma_{u2}=\left(2\gamma_{u1}^2+2\right)^{1/2}$, $\gamma_{u3}=(2\gamma_{u1}-\gamma_{u2})^{1/2}$ and $\gamma_{u4}=(2\gamma_{u1}+\gamma_{u2})^{1/2}$. By setting $\zeta=0$ in (22), the constant $A_u(0)$ is calculated directly as

$$\begin{aligned} A_u(0) = {} & \frac{-\pi}{2}+\left(\frac{1+\gamma_{u1}\left(\gamma_{u3}^2-\gamma_{u1}\right)}{2\gamma_{u1}\gamma_{u2}\gamma_{u3}}\right)\ln\left(\frac{1+\gamma_{u3}+\gamma_{u1}}{1-\gamma_{u3}+\gamma_{u1}}\right) \\ & +\left(\frac{1+\gamma_{u1}\left(\gamma_{u4}^2-\gamma_{u1}\right)}{2\gamma_{u1}\gamma_{u2}\gamma_{u4}}\right)\ln\left(\frac{1-\gamma_{u4}+\gamma_{u1}}{1+\gamma_{u4}+\gamma_{u1}}\right)+\left(\frac{3\gamma_{u1}^2-\gamma_{u1}\gamma_{u3}^2+1}{\gamma_{u1}\gamma_{u2}\gamma_{u4}}\right) \\ & \times\left[\arctan\left(\frac{2+\gamma_{u3}}{\gamma_{u4}}\right)+\arctan\left(\frac{2-\gamma_{u3}}{\gamma_{u4}}\right)\right]-\left(\frac{3\gamma_{u1}^2-\gamma_{u1}\gamma_{u4}^2+1}{\gamma_{u1}\gamma_{u2}\gamma_{u3}}\right) \\ & \times\left[\arctan\left(\frac{2+\gamma_{u4}}{\gamma_{u3}}\right)+\arctan\left(\frac{2-\gamma_{u4}}{\gamma_{u3}}\right)\right]+\ln(8). \end{aligned} \tag{23}$$

Similarly, for the temperature profile, the solution for integral $A_t(\zeta)$ yields

$$A_t(\zeta) = \ln(y+1)^2 + \left(\frac{\gamma_{t1}+1}{2\gamma_{t1}\gamma_{t2}}\right)\ln\left(\frac{y^2+\gamma_{t2}y+\gamma_{t1}}{y^2-\gamma_{t2}y+\gamma_{t1}}\right) + \left(\frac{1-\gamma_{t1}^{-1}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right) \times \left[\arctan\left(\frac{2y+\gamma_{t2}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right) + \arctan\left(\frac{2y+\gamma_{t2}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right)\right], \tag{24}$$

where $y = (1-\gamma_t\zeta)^{1/2}$, $\gamma_{t1} = \left(1+\gamma_t^2c^2\right)^{1/2}$ and $\gamma_{t2} = (2+2\gamma_{t1})^{1/2}$. As in the previous case, the constant $A_t(0)$ becomes

$$A_t(0) = \ln(4) + \left(\frac{\gamma_{t1}+1}{2\gamma_{t1}\gamma_{t2}}\right)\ln\left(\frac{1+\gamma_{t2}+\gamma_{t1}}{1-\gamma_{t2}+\gamma_{t1}}\right) + \left(\frac{1-\gamma_{t1}^{-1}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right) \times \left[\arctan\left(\frac{2+\gamma_{t2}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right) + \arctan\left(\frac{2-\gamma_{t2}}{\sqrt{4\gamma_{t1}-\gamma_{t2}^2}}\right)\right]. \tag{25}$$

In the case of integral $B_{u,t}(\zeta)$,the general solution is the same for both the wind speed and temperature profiles. However, two choices for the type of the convective form have been discussed here, namely Equations (5) and (7). If the first form is used in (21), the integral $B_{u,t}(\zeta)$ (called $B_{1u,t}(\zeta)$) results in the following simple expression

$$B_{1u,t}(\zeta) = \frac{-c^{-1/3}}{2\alpha_{\mathrm{C}u,t}^{1/3}}\left[\arctan\left(\frac{-3(-\zeta/c)^{1/3}\left((-\zeta/c)^{2/3}-1\right)}{\left((-\zeta/c)^{2/3}-1\right)^2-2(-\zeta/c)^{2/3}}\right) + \frac{-\sqrt{3}}{2}\ln\left(\frac{\left(\sqrt{3}+2(-\zeta/c)^{1/3}\right)^2+1}{\left(\sqrt{3}-2(-\zeta/c)^{1/3}\right)^2+1}\right)\right]. \tag{26}$$

From (26) it follows that, in this case, the constant $B_{1u,t}(0)$ is always zero, independent of the free parameters, $\alpha_{\mathrm{C}u,t}$, $\gamma_{u,t}$ and c:

$$B_{1u,t}(0) = 0. \tag{27}$$

If the convective forms given by (7) are used instead, the analytical solution for $B_{u,t}(\zeta)$ (called $B_{2u,t}(\zeta)$) yields the following, more complicated expression:

$$
\begin{aligned}
B_{2u,t}(\zeta) = {} & \frac{-\alpha_{\mathrm{R}}}{4|\alpha|^2}\ln\left(\frac{\left(z^4-2\alpha_{\mathrm{R}}z^3+(3\alpha_{\mathrm{R}}^2-\alpha_{\mathrm{I}}^2)z^2-2\alpha_{\mathrm{R}}|\alpha|^2z+|\alpha|^4\right.}{\left(z^2+2\alpha_{\mathrm{R}}z+|\alpha|^2\right)^2}\right)\\
& +\frac{-\sqrt{3}\alpha_{\mathrm{R}}}{2|\alpha|^2}\arctan\left(\frac{\sqrt{3}\alpha_{\mathrm{R}}z-2\sqrt{3}|\alpha|^2}{2|\alpha|^2-4z^2+4\alpha_{\mathrm{R}}z}\right)\\
& +\frac{\sqrt{3}\alpha_{\mathrm{I}}}{4|\alpha|^2}\ln\left(\frac{\left(\sqrt{3}\alpha_{\mathrm{R}}+\alpha_{\mathrm{I}}\right)^2+\left(2z+\sqrt{3}\alpha_{\mathrm{I}}-\alpha_{\mathrm{R}}\right)^2}{\left(\sqrt{3}\alpha_{\mathrm{R}}-\alpha_{\mathrm{I}}\right)^2+\left(2z-\sqrt{3}\alpha_{\mathrm{I}}-\alpha_{\mathrm{R}}\right)^2}\right)\\
& +\frac{\alpha_{\mathrm{I}}}{2|\alpha|^2}\arctan\left(\frac{3\alpha_{\mathrm{I}}z^3-3\alpha_{\mathrm{I}}|\alpha|^2z}{\left(z^2+2\alpha_{\mathrm{R}}z+|\alpha|^2\right)^2-6|\alpha|^2z^2-3\alpha_{\mathrm{R}}z^3-3\alpha_{\mathrm{R}}|\alpha|^2z}\right),
\end{aligned}
\tag{28}
$$

where $z=\left(1-\alpha_{\mathrm{C}u,t}\zeta\right)^{1/3}$, and α_{R} and α_{I} represent, respectively, the real and the imaginary part of the complex number $\left(-1+\mathrm{i}\alpha_{\mathrm{C}u,t}c\right)^{1/3}$ and $|\alpha|=\left(\alpha_{\mathrm{R}}^2+\alpha_{\mathrm{I}}^2\right)^{1/2}$. Then, the constant $B_{2u,t}(0)$ results in:

$$
\begin{aligned}
B_{2u,t}(0) = {} & \frac{\sqrt{3}\alpha_{\mathrm{I}}}{4|\alpha|^2}\ln\left(\frac{\left(\sqrt{3}\alpha_{\mathrm{R}}+\alpha_{\mathrm{I}}\right)^2+\left(2+\sqrt{3}\alpha_{\mathrm{I}}-\alpha_{\mathrm{R}}\right)^2}{\left(\sqrt{3}\alpha_{\mathrm{R}}-\alpha_{\mathrm{I}}\right)^2+\left(2-\sqrt{3}\alpha_{\mathrm{I}}-\alpha_{\mathrm{R}}\right)^2}\right)\\
& +\frac{-\alpha_{\mathrm{R}}}{4|\alpha|^2}\ln\left(\frac{1-2\alpha_{\mathrm{R}}+(3\alpha_{\mathrm{R}}^2-\alpha_{\mathrm{I}}^2)-2\alpha_{\mathrm{R}}|\alpha|^2+|\alpha|^4}{\left(1+2\alpha_{\mathrm{R}}+|\alpha|^2\right)^2}\right)\\
& +\frac{-\sqrt{3}\alpha_R}{2|\alpha|^2}\arctan\left(\frac{\sqrt{3}\alpha_{\mathrm{R}}-2\sqrt{3}|\alpha|^2}{2|\alpha|^2-4+4\alpha_{\mathrm{R}}}\right)\\
& +\frac{\alpha_{\mathrm{I}}}{2|\alpha|^2}\arctan\left(\frac{3\alpha_{\mathrm{I}}-3\alpha_{\mathrm{I}}|\alpha|^2}{\left(1+2\alpha_{\mathrm{R}}+|\alpha|^2\right)^2-6|\alpha|^2-3\alpha_{\mathrm{R}}-3\alpha_{\mathrm{R}}|\alpha|^2}\right).
\end{aligned}
\tag{29}
$$

In Figure 2, the results of Equation (19), for both wind speed and temperature functions, are illustrated for a variety of $\alpha_{\mathrm{C}u,t}$ values. The calculations refer to the solution for $B_{u,t}(\zeta)$ given by (28), with $\gamma_{u,t}=16$ and $c=1$. As expected, the functions have a smooth, physically acceptable behaviour; this applies also to the case of the simpler Equation (26) that produces almost identical curves. The corresponding results from (14) are also presented in the same Figure 2.

From the comparison follows that the results of (14) coincide (within $\pm 1\%$) with the new approach only for the specific choice $\alpha_{\mathrm{C}u}=10$ and

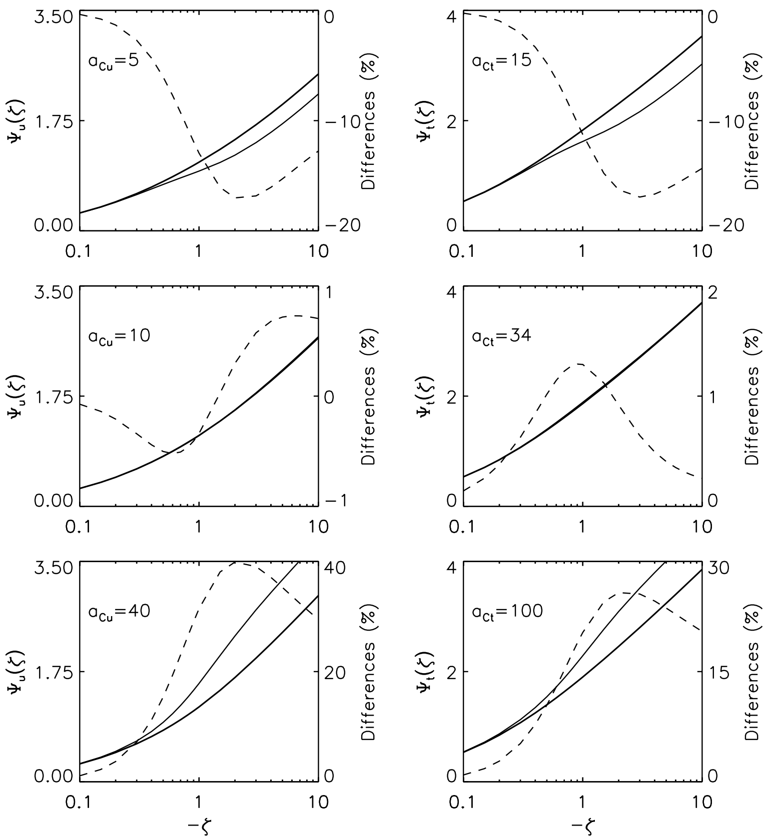

Figure 2. $\Psi_{u,t}(\zeta)$, resulting from the new Equation (17) (continuous heavy lines) and (14) (continuous thin lines), for different choices of parameters $\alpha_{Cu,t}$ with $\gamma_{u,t} = 16$ and $c = 1$. Their relative difference (%) (dashed lines; right axis) is also illustrated, for a quantitative comparison.

$\alpha_{Ct} = 34$. For other choices of the convective constants there is strong deviation between the two methods, which appears as an almost parallel shift of the convective part of (14) for $-\zeta > 1$. This behaviour suggested the idea of fitting the complicated Equations (19)–(29) with a much simpler form, through a modification of (14). After evaluating several approaches, it was concluded that an excellent approximation obtains by a simple vertical and horizontal translation of the convective form, $\Psi_{Cu,t}(\zeta)$ given by (12), in (14). Incorporating this translation, (14) becomes:

$$\Psi_{u,t}(\zeta) = \frac{c^2\Psi_{Ku,t}(\zeta) + \zeta^2\big(\Psi_{Cu,t}(\zeta + \zeta_\alpha) + \Delta(c)\big)}{c^2 + \zeta^2}. \tag{30}$$

In (30) the correcting factor $\Delta(c)$ is the vertical difference between $\Psi_{Ku,t}(\zeta)$, given by (10) or (11) for the wind speed and temperature, respectively, and $\Psi_{Cu,t}(\zeta)$, given by (12), at $-\zeta = c$:

$$\Delta(c) = \Psi_{Ku,t}(c) - \Psi_{Cu,t}(c). \tag{31}$$

The value ζ_α for the horizontal translation is calculated through

$$\Psi_{Cu,t}(\zeta_\alpha) = -\Delta(c), \tag{32}$$

so that the translated convective part, $\Psi_{Cu,t}(\zeta + \zeta_\alpha) + \Delta(c)$, returns 0 at the neutral limit. The last translation is restricted by the fact that the maximum value for ζ_α must be less than $1/\alpha_{Cu,t}$, to ensure $(1 - \alpha_{Cu,t}\zeta_\alpha) > 0$. However, this limitation was found to be less restrictive than the demand imposed upon (17) to have continuous and monotonic first derivative. For $\gamma_{u,t} = 16$ and $c = 1$, (30) differs from the original exact solution (19)–(29) by less than $\pm 1\%$, in the ranges $3 < \alpha_{Cu} < 40$ and $15 < \alpha_{Ct} < 100$.

5. Discussion

The interpolation according to (17) provided functions that permit quite a wide range for the values of convective constants $\alpha_{Cu,t}$. Unlike (14), the behaviour of the convective forms $\phi_C(\zeta)$, given by (5) or (7), at low instabilities does not affect the results markedly. This differentiation is important because the use of convective forms could prove inadequate for many applications. This is due to the, possibly, poor fitting of the convective functions ϕ_C near the neutral stability limit, which affects the results strongly in terms of $\Psi_C(\zeta)$ forms. In other words, the integration of $\phi_C(\zeta)$, via (8) in the range [0, $\zeta \sim 1$], overestimates or underestimates the respective results of the Kansas-type functions. This deviation is maintained and distorts the results for $\Psi_C(\zeta)$, although the behaviour of $\phi_C(\zeta)$ may be appropriate at the free convection limit.

The functions that have been investigated are intended for use, mostly, in the calculation of surface fluxes. In order to examine the impact of the above forms, wind speed and temperature profiles have been produced. More specifically, the profiles of the dimensionless wind speed, $kU(\zeta)/u_*$, and temperature, $k\Delta\theta(\zeta)/\theta_*$, are illustrated in Figure 3 for different values of the convective constants $\alpha_{Cu,t}$. The calculations have been done for the values of the dimensionless roughness length, ζ_0, 0.01 and 1. The first is a low value that corresponds to smooth surfaces for low or moderate instability. For this choice, it becomes clear that the profiles produced through (19) coincide with the established Kansas estimates for low instability and then follow a convective shape. Despite the large variation of $\alpha_{Cu,t}$, all the resulting profiles have a smooth shape. For $\alpha_{Cu} = 5$ and $\alpha_{Ct} = 100$ the profiles are almost identical to the Kansas ones, while for $\alpha_{Cu} = 10$ and $\alpha_{Ct} = 34$ they almost coincide with the convective profiles resulting from (12), since for this choice of the convective constants the differences between $\Psi_C(\zeta)$ and $\Psi_K(\zeta)$

diminish for $-\zeta < 1$. In fact, this is the reason for Grachev et al. (2000) choosing that specific set of values for their interpolation applied directly to the Ψ functions in (14). However, it must be pointed out that the relative differences among all profiles produced in this work by using $\alpha_{Cu} > 5$ and $\alpha_{Ct} > 15$ are about 5% (low sensitivity). This is attributed to the fact that the largest portion of the profile's change occurs at the lower instability range, where the Kansas-type shape has been kept constant, independent of the

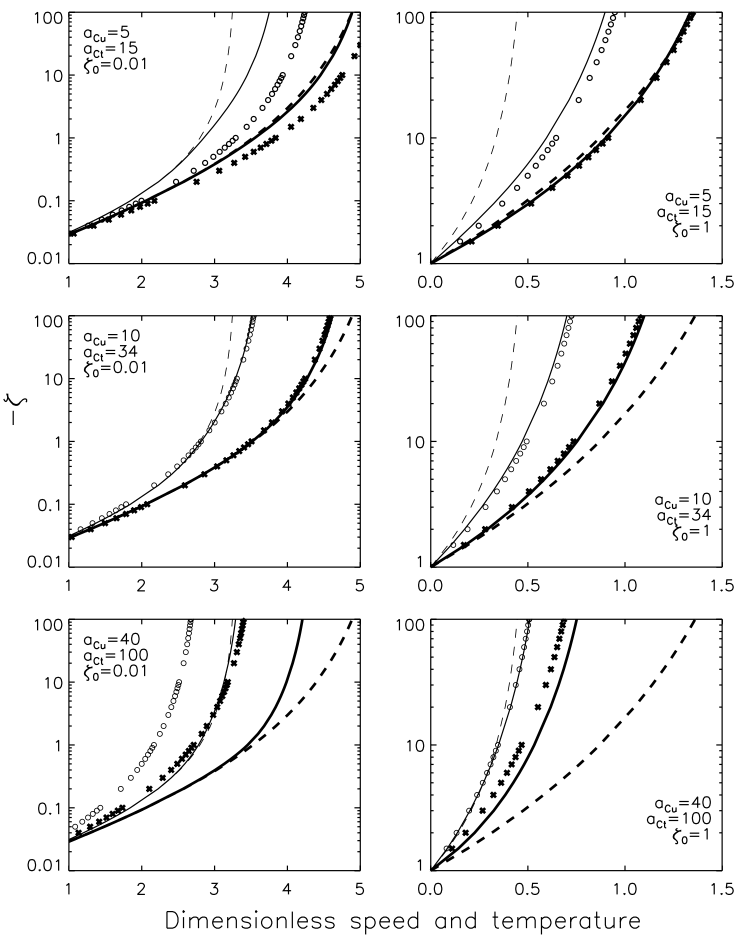

Figure 3. Profiles of the dimensionless wind speed, $kU(\zeta)/u_*$, and temperature, $k\Delta\theta(\zeta)/\theta_*$, for different values of the convective constants $\alpha_{Cu,t}$, and for $\zeta_{0u,t}$ equal to 0.01 and 1. The wind speed profiles have been calculated through (9) by using the definitions for $\Psi_u(\zeta)$, given by (10) (heavy dashed lines), (12) (crosses) and (19) (heavy continuous lines). The temperature profiles have been calculated through (9) by using the definitions for $\Psi_t(\zeta)$, given by (11) (thin dashed lines), (12) (open circles) and (19) (continuous thin lines).

values of the convective parameters. On the other hand, the convective profiles based on (12) show a strong dependence on the value of the convective constants, as would be expected. The influence of this dependence extends to the near-neutral regime, therefore such forms should be used only when the profile departs from very large ζ_0 values. This is corroborated also from the investigation of the profiles that depart from $\zeta_0 = 1$; these profiles thus refer to a development over rough surfaces during strong instability. For such cases the shape of the profile for low values of ζ loses its importance. As a result, the interpolation introduced by (17) leads to almost the same vertical distribution as the convective form (7). Additionally, the dependence of the profile on the choice of $\alpha_{Cu,t}$ becomes profound, resulting in a relative difference between each choice greater than 25%. Larger values of those constants result in lower values of dimensionless wind speed and more efficient mixing (larger values for u_*) during low wind speeds and strong instability. This behaviour may prove to be very important in the flux calculation and in the 'minimum friction velocity' for free convection conditions (Businger, 1973; Akylas et al., 2001).

The above results clarify the importance of a correct estimation of the convective constants through the fitting of the theoretical forms (either Equations (5) or (7)) on experimental data. The fitting should apply to strong instability, where these types are most suitable. For lower instabilities, however, Kansas-types are a generally accepted representation. The importance of which part of the unstable regime is used, in order to estimate the convective constants, can be shown from the following examples that use experimental measurements.

In Figures 4a and b measurements published by Carl et al. (1973) and Businger et al. (1971) are illustrated, showing values of $\phi_u(\zeta)$ and $\phi_t(\zeta)$, respectively. In the first figure, Carl et al. (1973) obtained $\alpha_{Cu} = 16$ by fitting (7) to the whole dataset. If the fitting is applied just to the range $-\zeta > 1$, the previous value should be modified to $\alpha_{Cu} = 30$. In that case, the interpolation given by (17) describes the whole range very well. In Figure 4b, fitting (7) to the whole dataset results in a convective constant, $\alpha_{Ct} = 65$, while focussing on data with $-\zeta > 1$, leads to an enhanced value of $\alpha_{Ct} = 100$. A good overall behaviour is achieved by using the second value in the interpolation according to (17).

It is noted here that the previous simplified examples are not intended for the deduction of particular values of the convective constants, but only to outline the importance of a suitable functional form for fitting experimental data. Thus, possible problems with the need for corrections concerning the presented measurements (Wieringa, 1980) have not been taken into account in this work. For the evaluation of exact values, experimental data of high quality under very strong instability are therefore required. This is a challenging task for further research.

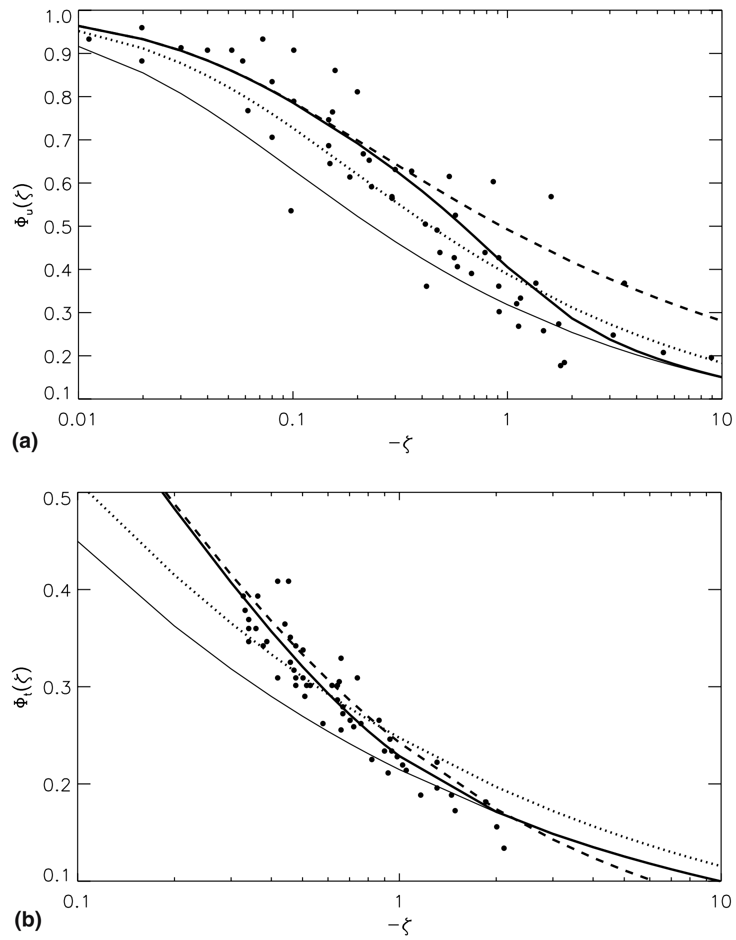

Figure 4. (a) Measurements of $\phi_u(\zeta)$, based on experimental data (points) by Carl et al. (1973), as well as, (7) for $\alpha_{Cu} = 16$ (dotted line), (6) for $\gamma_u = 16$ (dashed line), (7) for $\alpha_{Cu} = 30$ (continuous thin line) and their interpolation, given by (17) (continuous heavy line). (b) Measurements of $\phi_t(\zeta)$, based on Kansas experimental data (points) by Businger et al. (1971), as well as, (7) for $\alpha_{Ct} = 65$ (dotted line), (6) for $\gamma_t = 16$ (dashed line), (7) for $\alpha_{Ct} = 100$ (continuous thin line) and their interpolation, given by (17) (continuous heavy line).

6. Conclusions

We have shown in this study that the possible poor fitting of the convective gradient relations, $\phi_{Cu,t}(\zeta)$, at near-neutral stability, affects the results markedly in terms of $\Psi_{Cu,t}(\zeta)$ functions and the corresponding vertical profiles. Such functions could be used only over very rough surfaces during strong instability due to the large value of $\zeta_0\,(\sim 1)$ that eliminates the near neutral dependence, since the integration starts from large values of ζ. The choice of values of the convective constants $\alpha_{Cu,t}$ becomes then very important for the profile shape and the flux calculations.

Unlike previous work, which has focused on flux–profile relations, an interpolation between Kansas-type and convective forms has been applied directly to the relevant gradient functions, in order to produce functions appropriate for the whole unstable range. Exact expressions have been derived for the Ψ forms through analytical quadrature. The new treatment results in smooth and monotonic functions for a wider range of $\alpha_{Cu,t}$ choices. Thus, the appropriate values of these constants could be determined by comparison with experimental data. Using the derived Ψ forms as standards, a simple interpolation procedure has been elaborated that is based on the established Kansas-type and convective flux–profile relations. The proposed efficient interpolation procedure affords an accuracy of $\pm 1\%$ and can be used in modelling applications, covering every case with $\zeta < 0$.

Acknowledgements

The authors would like to express their special thanks to Dr. A. Koussis, Dr. E. Defer, Dr. A. Giannakopoulos, Dr. P. Good, and to two anonymous referees for useful discussions and comments.

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