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Analytical solution of transient flow in a sloping soil layer with recharge

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Abstract An analytical solution of planar flow in a sloping soil layer described by the linearized extended Boussinesq equation is presented. The solution consists of the sum of steady-state and transient-series solutions, the latter in a separation-of-variables form, and can satisfy an arbitrary initial condition via collocation; this feature reduces the number of series terms, making the solution efficient. Key parameter is the dimensionless linearization depth $\eta_o(R)$, R being the dimensionless recharge. The variable $\eta_o(R)$, not the slope, characterizes the flow as kinematic or diffusive, and $R \approx 0.2$ demarcates the two regimes. The transient series converges rapidly for large η_o (large R, near-diffusive flow) and slowly as $\eta_o \to 0$ (kinematic flow). The quasi-steady (QS) state method of Verhoest & Troch is also analysed and it is shown that the QS depth profiles approximate the transient ones well, only if Δt exceeds a system-dependent transition time between flow states (possibly >>1 day). In an application example for a 30-day recharge series, the QS solution differs from the transient one by as much as 20% ($RMSE$ = 15%), does not track recharge changes as well and fails to conserve mass.

Key words analytical solution; hillslope flow; quasi-steady flow; recharge; subsurface stormflow

Solution analytique d'un écoulement transitoire dans une couche de sol en pente avec recharge

Résumé Nous présentons une solution analytique d'un écoulement planaire dans une couche de sol en pente décrit par l'équation de Boussinesq linéarisée. La solution est la somme d'une composante continue et d'une série de transitoires, ces dernières étant exprimées sous forme de séparation des variables. Cette solution peut satisfaire une condition initiale arbitraire par collocation, ce qui diminue le nombre de transitoires et fournit une solution plus performante. Le paramètre clef est la profondeur de linéarisation $\eta_o(R)$, adimensionnelle, en fonction du paramètre adimensionnel de recharge R. La variable $\eta_o(R)$, et non la pente, détermine le caractère cinématique ou diffusif de l'écoulement, et $R \approx 0.2$ marque la transition entre les deux régimes. La série de transitoires converge rapidement pour les fortes valeurs de η_o (pour R important, écoulement presque diffusif) mais converge lentement lorsque $\eta_o \to 0$ (écoulement cinématique). Nous analysons également la méthode de l'état quasi-continu (QC) de Verhoest & Troch et montrons que les profils de profondeur de QC se rapprochent de ceux des transitoires seulement si Δt dépasse un temps de transition, dépendant du système, entre les différents états de l'écoulement (potentiellement >> 1 jour). En prenant pour exemple une série de recharge de 30 jours, la solution QC diffère de celle des transitoires de plus de 20% ($RMSE = 15\%$), ne suit pas les changements de recharge et échoue à conserver la masse.

Mots clefs solution analytique; écoulement de versant; écoulement quasi-continu; recharge; écoulement événementiel de subsurface

NOTATION

The notation y, Y = … refers to a dimensional and a non-dimensional quantity, respectively.

A, B constants in the steady-state solution;

C constant in the time-dependent function of the transient solution;

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c_1, c_2 and c_{1i}, c_{2i} coefficients in the series of the transient solution;
$F(X, T)$ transient part of the solution, postulated in product form = $\phi(X)\ \psi(T)$;
$G(X)$ steady-state part of the solution;
h, H flow depth measured as height of water column normal to the bed;
h_o, H_o linearization depth of flow in the porous layer;
K hydraulic conductivity of porous layer;
L length of porous layer;
$m_{\alpha,\beta}$ roots of characteristic equation of spatial differential equation;
n drainable porosity (specific yield) of porous layer;
q, Q discharge per unit width;
r, R recharge rate;
r^* recharge rate used in the quasi-steady-state solution;
$S = \sin\varphi$ slope of soil layer's base;
t, T time;
x, X distance measured along the soil layer's base;
η_o linearization parameter = $H_o \cos\varphi$;
λ, λ_i parameters in exponential function of transient series solution;
μ, μ_i parameters in trigonometric spatial series in transient solution;
$\rho = r/K$ ratio of recharge rate to hydraulic conductivity;
$\sigma = S(1 - \rho)$ reduced slope;
τ duration of a recharge pulse; and
φ angle of inclination of soil layer base.

INTRODUCTION

Flow through a soil layer resting on an inclined bed is essential for the hydrology of upland watersheds (storm or baseflow) and is related to the interflow; it also occurs in landfills as lateral flow to leachate collection drains installed above liners. Recharge (rate r per unit horizontal area) infiltrating through the soil accumulates over a low-conductivity bed forming a saturated flow layer, as shown in Fig. 1. In the hydraulic description of unconfined flow through a porous medium of hydraulic conductivity K and drainable porosity n (specific yield), known as the Dupuit-Forchheimer theory, the pressure in the water column of height h normal to the bed is taken as hydrostatic. Accordingly, discharge per unit width (planar flow), at time t and at location x, measured from the top of hill along the inclined base of length L and angle φ against the horizontal ($S = \sin\varphi$), is (e.g. Wooding & Chapman, 1966):

$$q(x,t) = h(KS - K\frac{\partial h}{\partial x}\cos\varphi) \tag{1}$$

Combining equation (1) with the storage balance equation (Wooding & Chapman, 1966):

$$n\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r\cos\varphi + r\frac{\partial h}{\partial x}\sin\varphi \tag{2}$$

(the right-hand side derives from the scalar product of the recharge vector and the unit

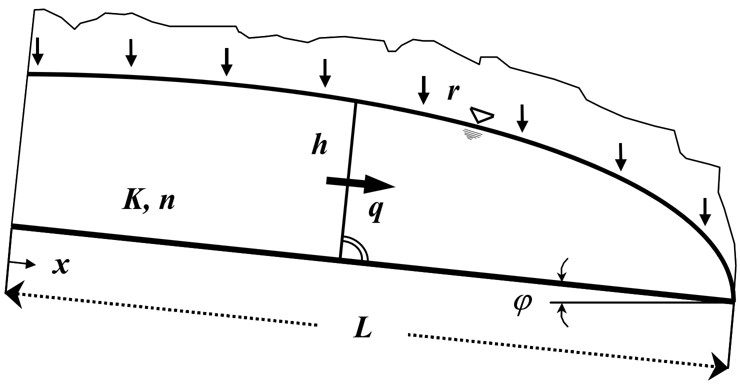

Fig. 1 Cross-sectional schematic diagram of sloping soil layer, with definition of symbols.

normal of a free surface element), yields the extended Boussinesq (1877) equation of unconfined flow over an inclined base:

$$n\frac{\partial h}{\partial t}+KS\left(1-\frac{r}{K}\right)\frac{\partial h}{\partial x}-\cos\varphi K\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)=r\cos\varphi \qquad (3)$$

Chapman (2005) gives an excellent review of the governing equations, also comparing the solution of equation (3) to those of its linearized form. Henderson & Wooding (1964) neglected the last term in equation (2) and obtained equation (3) without the factor $(1 - r/K)$. That equation has been adopted widely (e.g. Beven, 1981; Koussis & Lien, 1982; Koussis, 1992; Koussis *et al*., 1998; Verhoest & Troch, 2000; Pauwels *et al*., 2002; Basha & Maalouf, 2005) and holds for $r/K \ll 1$ (e.g. for $r = 4$ mm h^{-1} and $K = 10$ m day^{-1}, $r/K \approx 0.01$). Chapman (2005) showed that the mean steady flow depths, computed with and without the r/K term in equation (3), vary appreciably only for very large r/K values, $r/K \geq 0.1$, say. Given that typically $r/K < 0.05$ and the uncertainty in the values of K and r, the approximation seems justified in many cases.

Based on the soil moisture storage concept of Fan & Bras (1998), Troch *et al.* (2003) developed and solved numerically subsurface stormflow models for a variable-width hillslope; for constant width, these reduce to equation (3) (and upon linearization, to equation (5) below). As Brutsaert (1994) points out, the schematization of the physical situation and Dupuit's hydraulic treatment are trade-offs that enable parameterization of solutions for inclusion in complex models (Pauwels *et al*., 2002). The nonlinearity of equation (3) necessitates additional simplifications to allow analytical solutions. Of these, the most drastic is the kinematic wave (KW) approximation, proposed by Boussinesq (1877) and evaluated by Henderson & Wooding (1964) and by Beven (1981). The KW model postulates $S \gg |\partial h/\partial x|$ (clearly failing as $\varphi \to 0$), whence $q \approx hKS$ and the term $\cos\varphi K\partial(h\partial h/\partial x)/\partial x$ in equation (3) is eliminated.

The Linear Advection-Dispersion (LAD) model, introduced by Koussis & Lien (1982) and studied by Koussis (1992), Brutsaert (1994), Koussis *et al.* (1998), Verhoest & Troch (2000) and Pauwels *et al.* (2002), among others, gives a closer approximation. It admits that gravity dominates the net pressure force, but retains the effect of the surface gradient in a linearized term of hydraulic diffusion. The LAD governing equations are:

$$q = h\,KS - h_o K\cos\varphi \frac{\partial h}{\partial x} \tag{4}$$

$$n\frac{\partial h}{\partial t} + K\,S\left(1-\frac{r}{K}\right)\frac{\partial h}{\partial x} = Kh_o\cos\varphi\frac{\partial^2 h}{\partial x^2} + r\cos\varphi \tag{5}$$

The linearization depth h_o can be estimated, e.g., as fraction of the mean steady flow depth on the hillslope for recharge r (Koussis, 1992) (see equation (35) below).

Brutsaert (1994) derived an analytical LAD solution for drainage of an initially stagnant, uniform water block, which Verhoest & Troch (2000) extended to include recharge. This initial stagnant uniform depth is, however, unrealistic, as uniform depth with a free surface is not realizable in a finite domain; also, a free surface parallel to the inclined bed conflicts with the no-flow up-gradient boundary condition that prescribes a horizontal water table (vanishing hydraulic gradient) (Koussis *et al*., 1998). It is this unphysical stagnant water block that causes the outflow oscillation after initiation of drainage. The water column near the outlet supplies the large outflow rates at t = 0+, generating steep gradients there; however, this condition cannot be maintained and subsequent outflows are more gravity-driven. For this reason, on steep slopes, the outflow can even rise slightly after an initial decline.

Chapman's (1995) adaptation of Werner's (1957) analytical solution for a recharge step starts from steady flow and reflects reality better; it is based on the h^2-linearization of equation (3). Brutsaert (1995) argues in favour of the h-linearization because of its compatibility with the KW model for $S >> |\partial h/\partial x|$. Basha & Maalouf (2005) examine both linearizations. Verhoest & Troch (2000) solve equation (5), with $r/K = 0$, also for a steady initial flow and show that the soil layer's response differs markedly from that for a stagnant initial uniform depth; for efficient calculations, they develop a quasi-steady-state method (Polubarinova-Kochina, 1962).

The focus of this work is on an efficient analytical solution of the problem. Such a solution is presented for a recharge step that holds for arbitrary known initial conditions (t = 0) such as a dry bed, a steady or a transient flow. Results are generalized by non-dimensionalization. The response to a finite-duration recharge pulse is obtained by combining build-up and recession solutions. Finally, a quasi-steady solution and the new analytical solution are compared in an application example, in which the response to a recharge series is given by superposition of properly timed individual pulse responses.

NON-DIMENSIONALIZATION AND INITIAL AND BOUNDARY CONDITIONS

Introducing the abbreviations $\rho = r/K$, $\sigma = S(1 - \rho)$, the variables are normalized as follows:

$$X = x/L;\ H = h/L\sigma;\ H_o = h_o/L\sigma;\ T = tK\sigma/nL;\ R = r\cos\varphi/K\sigma^2;\ \text{and}\ Q = q/r\cos\varphi L \tag{6}$$

The dimensionless, inhomogeneous LAD equation governing the depth becomes thus:

$$\frac{\partial H}{\partial T} + \frac{\partial H}{\partial X} = H_o\cos\varphi\frac{\partial^2 H}{\partial X^2} + R \tag{7}$$

where $H_o = h_o/L\sigma$. The corresponding non-dimensional form of the discharge equation (4) reads:

$$RQ = H/(1-\rho) - H_o\cos\varphi\,\partial H/\partial X \tag{8}$$

As initial condition, the bed is assumed to be dry:

$$h(x,0) = H(X,0) = 0 \tag{9}$$

At the upper boundary, no flow is assumed, $q(0,t) = Q(0,T) = 0$, whence equations (4) and (8) yield:

$$\cos\varphi\,\partial h/\partial x|_o = Sh(0,t)/h_o \quad \text{and} \quad \cos\varphi\,\partial H/\partial X|_o = H(0,T)/(1-\rho)H_o \tag{10}$$

respectively (implies a solid barrier at right angle to the bed, as the depth is measured normal to the bed). This condition will be discussed after the development of the analytical solution. At the lower boundary $x = L(X = 1)$, free drainage is assumed:

$$h(L,t) = H(1,T) = 0 \tag{11}$$

$(H_o\cos\varphi)^{-1}$ is the cardinal parameter of the problem and a characteristic number of non-dimensional transport equations such as equation (7): this is the Reynolds number of momentum transport and the Péclet number of mass and heat transport (e.g. Tennekes & Lumley, 1972). Henceforth the compact notation $\eta_o = H_o\cos\varphi = h_o\cos\varphi/LS(1-\rho)$ is adopted.

DEVELOPMENT OF GENERAL ANALYTICAL SOLUTION

It is postulated that $H(X,T)$ is configured as follows:

$$H(X,T) = G(X) + F(X,T) = F(X) = \phi(X)\psi(T) \tag{12}$$

where $G(X)$ is the steady-state solution of the problem. Thus, upon introducing equation (12) into equation (7), the principal equation is split in the steady-state equation (13) and the transient equation (14):

$$\frac{dG(X)}{dX} = \eta_o \frac{d^2G(X)}{dX^2} + R \tag{13}$$

$$\frac{\partial F(X,T)}{\partial T} + \frac{\partial F(X,T)}{\partial X} = \eta_o \frac{\partial^2 F(X,T)}{\partial X^2} \tag{14}$$

The solution of equation (13) is:

$$G(X) = \eta_o A \exp\left(\frac{X}{\eta_o}\right) + RX + B \tag{15}$$

with constants A and B determined by the boundary conditions equations (10)–(11) as follows:

$$A = -R\frac{1-\rho+\eta_o^{-1}}{\exp\eta_o^{-1}-\rho}\;;\; B = \eta_o R\left[(1-\rho) + \left(\frac{1-\rho+\eta_o^{-1}}{\exp\eta_o^{-1}-\rho}\right)\rho\right] \tag{16}$$

Then, combining of equations (12) and (14) leads to the dual relationships:

$$\frac{\eta_o \phi''(X) - \phi'(X)}{\phi(X)} = \frac{\dot{\psi}(T)}{\psi(T)} = \lambda \tag{17}$$

prime and dot indicating d/dX and d/dT, respectively; thus λ = const. and equation (17) separates in:

$$\dot{\psi}(T) = \lambda \psi(T) \tag{18a}$$

$$\eta_o \phi''(X) - \phi'(X) - \lambda \phi(X) = 0 \tag{18b}$$

The solution of equation (18a) is equation (19), with $\lambda < 0$ (asymptotic approach to a finite steady state):

$$\psi(T) = C \exp(\lambda T) \tag{19}$$

$\phi(X) = \exp(mX)$ is postulated as solution of equation (18b), which holds when m satisfies the quadratic equation (20), with possibly complex roots $m_{\alpha,\beta}$ ($j = (-1)^{1/2}$, the imaginary unit):

$$\eta_o m^2 - m \quad - \lambda = 0 \tag{20}$$

$$m_{\alpha,\beta} = \frac{1 \pm \sqrt{1 + 4\eta_o \lambda}}{2\eta_o} = \frac{1}{2\eta_o} \pm \frac{\sqrt{1 + 4\eta_o \lambda}}{2\eta_o} = \frac{1}{2\eta_o} \pm j\mu \tag{21}$$

$$\mu^2 = -(1 + 4\eta_o \lambda) / 4\eta_o^{\,2} \tag{22}$$

The solution of equation (18b) is:

$$\phi(X) = \exp\frac{X}{2\eta_o}\left(c_1 \cos \mu X + c_2 \sin \mu X\right) \tag{23}$$

Writing $C = aR$ and absorbing the common factor a in c_1 and c_2, equations (19) and (23) yield:

$$F(X,T) = \phi(X)\,\psi(T) = R \exp\frac{X}{2\eta_o}\left(c_1 \cos \mu X + c_2 \sin \mu X\right)\exp(\lambda T) \tag{24}$$

the superposing of which accommodates any initial condition in the solution of equation (14):

$$F(X,T) = R \exp\frac{X}{2\eta_o}\sum_i\left(c_{1i} \cos \mu_i X + c_{2i} \sin \mu_i X\right)\exp(\lambda_i T) \tag{25}$$

By equation (12), the sum of equations (25) and (15) (A and B from equation (16)) yields the solution of equation (7):

$$H(X,T) = \eta_o A \exp(\frac{X}{\eta_o}) + RX + B + R \exp\frac{X}{2\eta_o}\sum_i\left(c_{1i} \cos \mu_i X + c_{2i} \sin \mu_i X\right)\exp(\lambda_i T) \tag{26}$$

The number and the values of the coefficients c_{1i} and c_{2i} and of the parameters μ_i and λ_i remain to be determined through implementation of the initial and boundary conditions. With equation (26), boundary condition equation (10), $\partial H/\partial X|_o = H(0,T)/(1-\rho)\eta_o$, translates to:

$$\frac{1+\rho}{1-\rho}\sum_i \frac{c_{1i}}{2\eta_o}\exp(\lambda_i T) = \sum_i \mu_i c_{2i}\exp(\lambda_i T) \tag{27}$$

For equation (27) to be valid for every T, c_{1i} and c_{2i} must obey the relationship:

$$c_{1i}/c_{2i} = 2\eta_o\mu_i(1-\rho)/(1+\rho) \tag{28}$$

With relationship equation (28) and the convention $c_i = c_{1i}$, the general solution equation (26) simplifies to:

$$H(X,T) = \eta_o A\exp(\frac{X}{\eta_o}) + RX + B + R\exp\left(\frac{X}{2\eta_o}\right)\sum_i c_i\left(\cos\mu_i X + \frac{1+\rho}{1-\rho}\frac{\sin\mu_i X}{2\eta_o\mu_i}\right)\exp(\lambda_i T) \tag{29}$$

Further, implementation of the boundary condition $H(1,T) = 0$, equation (11), yields:

$$\sum_i c_i\left(\cos\mu_i X + \frac{1+\rho}{1-\rho}\frac{\sin\mu_i}{2\eta_o\mu_i}\right)\exp(\lambda_i T) = 0 \tag{30}$$

Since $c_i \neq 0$ and $\exp(\lambda_i T) \neq 0$, the term in parenthesis vanishes, yielding the implicit relation:

$$\mu_i = -\frac{1+\rho}{1-\rho}\tan\mu_i/2\eta_o \tag{31}$$

From equations (31) and (22) follows that $\lambda \leq -(4\eta_o)^{-1}$ and the $m_{\alpha,\beta}$ roots of equation (21) are complex.

To fully define $H(X,T)$, the coefficients c_i are determined such that equation (29), for $T = 0$, satisfies equation (9) (or any other initial condition) at k points. The procedure, whereby two functions are exactly matched at k freely selectable points, is called collocation and yields a system of k linear equations for k parameters c_i; k controls the desired accuracy (see next section).

With the $H(X,T)$ defined, the flow rate can be evaluated from equation (8), giving:

$$Q(X,T) = \frac{\eta_o A}{R}\frac{\rho}{1-\rho}\left(\exp(\frac{X}{\eta_o})-1\right) + \frac{X}{1-\rho} + \exp\left(\frac{X}{2\eta_o}\right)\sum_i c_i\left[\eta_o\mu_i + \left(\frac{1+\rho}{1-\rho}\right)^2\frac{1}{4\eta_o\mu_i}\right]\sin\mu_i X\exp(\lambda_i T) \tag{32}$$

The coefficients c_i can be also determined from the system of linear equations obtained by collocating equation (32), for $T = 0$, with the initial flow rate $Q(X,0) = 0$. The quantity of prime hydrological interest, the outflow at the hill base, is obtained for $X = 1$:

$$Q(1,T) = -\frac{\eta_o A}{R}\exp\left(\eta_o^{-1}\right) - \eta_o + \exp\left(\frac{1}{2\eta_o}\right)\sum_i c_i\left[\eta_o\mu_i + \left(\frac{1+\rho}{1-\rho}\right)^2\frac{1}{4\eta_o\mu_i}\right]\sin\mu_i\exp(\lambda_i T) \tag{33}$$

In summary, to use equation (29) and equation (32): (a) estimate $\eta_o(R)$ (see equation (35)), (b) calculate the μ_i values from equation (31) and, with these in equation (22), the corresponding λ_i values and (c) determine the coefficients c_i by collocating equation (29) or equation (32), for $T = 0$, and the initial condition at k points.

The solution cannot describe flow in a horizontal soil layer. For $S = 0$, equation (5) reduces to a diffusion equation in h/L, x/L, t/nKL, $R = r/K$, for which the separation-of-variables solution approach gives the closed-form expression, with steady-state part $(R/2H_o)(1 - X^2)$ and periodic transient part (m_i is imaginary and $\lambda_i/H_o = -\mu_i^2 = [(2i-1)\pi/2]^2$):

$$H(X,T) = \frac{R}{2H_o}\left[\left(1-X^2\right) + \frac{32}{\pi^3}\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)^3}\cos\left(\frac{2i-1}{2}\pi X\right)\exp\left(-\frac{(2i-1)^2}{4}\pi^2 T\right)\right] \tag{34}$$

DISCUSSION AND ASSESSMENT OF THE LAD SOLUTION

Relationships (29) and (32) reveal the linearization parameter η_o as cardinal for the problem, since from equation (31) $\mu(\eta_o)$ and from equation (22) $\lambda(\mu(\eta_o),\eta_o)$. At steady state, η_o is postulated as the mean flow depth, which is the integral of equation (15) over the normalized slope length, 1 (Koussis, 1992). Since the mean steady flow depths computed with and without the term $r\sin\varphi\partial h/\partial x$ in equation (2) differ little if $r/K \leq 0.1$, equation (16) is used with $r/K = 0$ to estimate η_o to obtain:

$$\eta_o(R) = \int_0^1 G(X)\mathrm{d}X \equiv <G(R)> = 0.5R + R\eta_o - R\eta_o(\eta_o + 1)\left[1 - \exp(-1/\eta_o)\right] \tag{35}$$

Ignoring $\exp(-1/\eta_o)$ (≤ 0.1 if $R \leq 2$) yields a quadratic, with solution $\eta_o(R) = [(1 + 2R^2)^{1/2} - 1]/2R$. The maximum error of this solution is under 1% for $R \leq 1$, rising to 6% at $R = 2$, relative to an iterative solution of equation (35) (evaluate $\exp(-1/\eta_o)$ approximately, then update the quadratic with the result of equation (35) and repeat). Figure 2 shows the agreement of the linear with the nonlinear steady-state solution to be generally good and expectedly better for low R values.

The linearization basis may be found by calibration (Brutsaert, 1994; Verhoest & Troch, 2000), yet it should be kept in mind that, according to equation (35), $\eta_o = <G(R)> \cos\varphi$ and thus it is not a constant parameter. The dependence $\eta_o(R)$ points to the flow's nonlinearity; if gravity dominates, $\eta_o \to 0$ and, of course, the linear KW solution holds. Given the determinant role of η_o in the solution behaviour and the $\eta_o(R)$ variability manifested in Table 1, η_o should be linked to the recharge, so that flows from $r(t)$ may be better tracked. To calculate transient flow in the case of an initially dry soil layer recharged with r_{step} ($t > 0$), $0.5h_o(r_{\text{step}})$ may be used as linearization depth, as done in the calculation of the depth profiles and outflow hydrographs shown in Figs 3 and 4.

It is widely recognized that kinematic flow implies steep slopes. However, a slope-referenced characterization holds largely, not strictly. The flow is controlled by $\eta_o(R)$, and $R \approx r\cos\varphi/KS^2$; thus, indeed, the slope influences R more than the hydraulic conductivity or the recharge rate, but not exclusively. Beven (1981) presents related results in his Figures 3 and 5. For example, in two hydrogeologically identical soils ($K_1 = K_2$) lying on different slopes, flow behaviour is the same if their R values are the same, i.e. $r_1/S_1^2 = r_2/S_2^2$, e.g. 1 mm h$^{-1}/0.1^2$ = 4 mm h$^{-1}/0.2^2$. Similarly, two soil

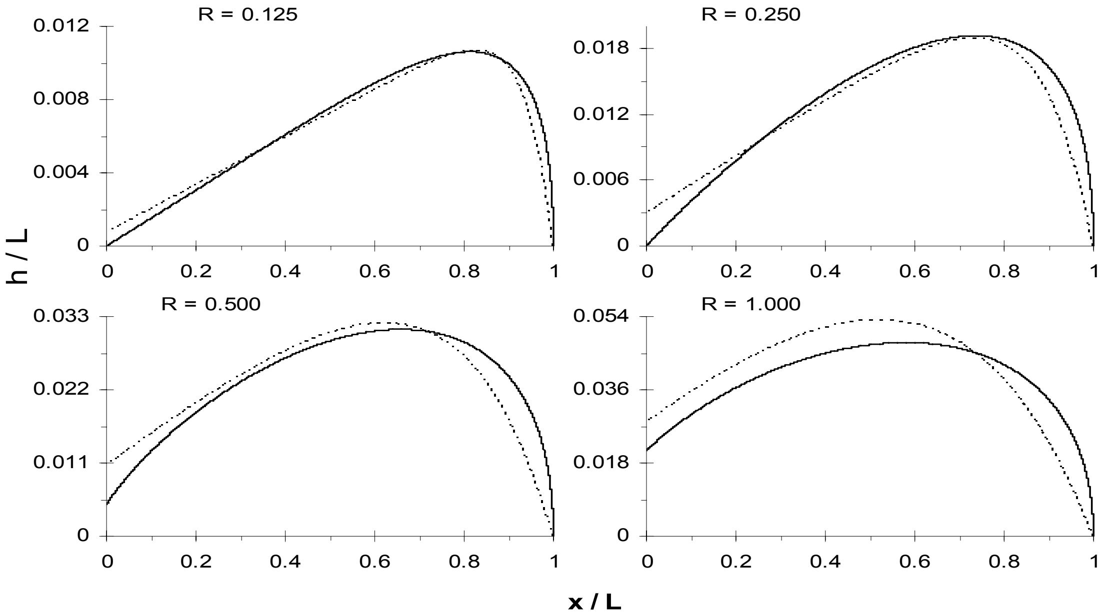

Fig. 2 Nonlinear (solid lines) and linear (dashed lines) depth profiles at steady state.

Table 1 Values of parameters μ_i and λ_i in series solutions equations (29) and (32).

R	η_o [a]	μ_1	μ_2	μ_3	λ_1	λ_2	λ_3
0.125	0.031	2.960	5.931	8.920	–8.332	–9.151	–10.528
0.250	0.062	2.810	5.675	8.612	–4.540	–6.040	–8.628
0.500	0.121	2.588	5.374	8.320	–2.880	–5.557	–10.425
0.750	0.176	2.442	5.220	8.195	–2.473	–6.196	–13.176
1.000	0.224	2.344	5.131	8.128	–2.346	–7.004	–15.885
2.000	0.364	2.157	4.992	8.031	–2.381	–9.758	–24.162

[a] 1/2 of the solution of equation (35) (assumes infinite duration recharge step).

layers lying on different slopes, but recharged equally, are hydraulically equivalent, provided $K_1S_1^2 = K_2S_2^2$, e.g. 10 m day^{-1} $(0.1)^2$ = 2.5 m day^{-1} $(0.2)^2$. Noteworthy is that the flow in the same medium is more kinematic or more diffusive depending on the recharge; e.g. with $S = 0.1$ and $K = 10$ m day^{-1}, $R = 0.24$ (near-kinematic) for r = 1 mm h^{-1} and $R = 0.96$ (diffusive) for r = 4 mm h^{-1}. An approximate kinematic-diffusive limit is $R = 0.2$.

The postponed discussion of zero flow at $x = 0$, $q(0, t) = 0$ will be now continued. The nonlinear flow relation equation (1) yields (in addition to $h = 0$) $\partial h/\partial x|_o = S/\cos\varphi = \tan\varphi$, indicating a horizontal water surface. In contrast, the linearized flow relationship equation (4) gives $\partial h/\partial x|_o = \tan\varphi h(0,t)/h_o$, i.e. an inclined water surface that deviates from the horizontal according to $h(0,t)/h_o$. Assessing $\partial h/\partial x|_o$ for steady flow by using equations (15)–(16) (with A and B for $\rho = r/K = 0$, to simplify the algebraic expressions) and after converting to dimensional quantities via equation (6), one obtains:

$$\partial h/\partial x|_o = \tan\varphi\{R[1 - (1 + \eta_o^{-1}) \exp(-\eta_o^{-1})] \cos\varphi\} \quad (36)$$

The ratio $\partial h/\partial x|_o/\tan\varphi$ depends mainly on R, since $\cos\varphi$ varies weakly (e.g. $0.866 \le \cos\varphi \le \sim 1.000$ in the range $1° \le \varphi \le 30°$). For $\eta_o \to 0$, $\partial h/\partial x|_o \to R\sin\varphi = r/K\sin\varphi$, as

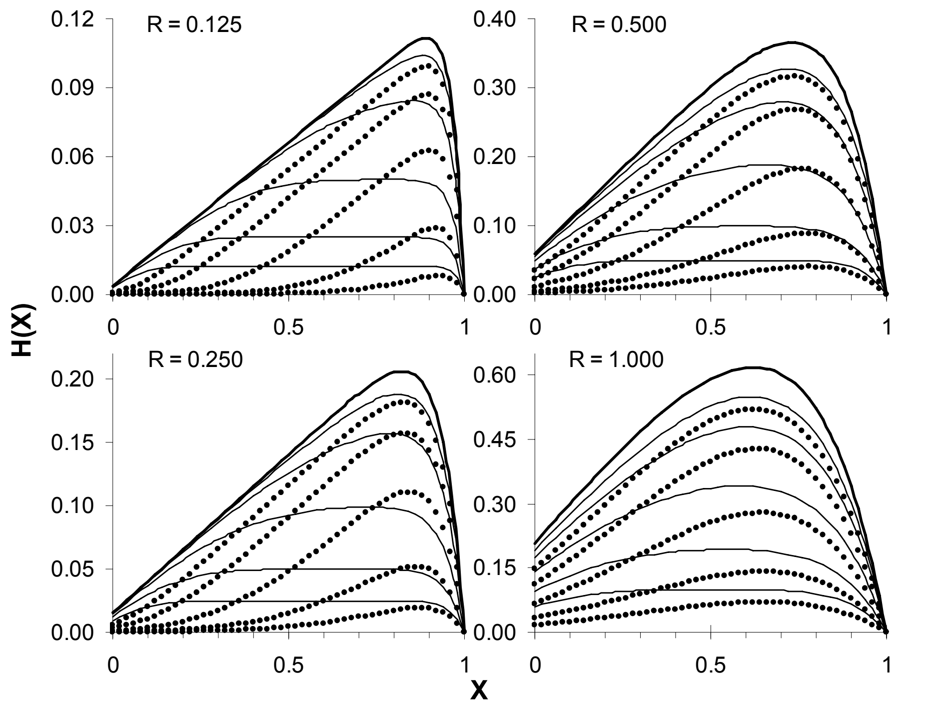

Fig. 3 Non-dimensional depth profiles $H(X,T)$: build-up (——) and drainage (•••) phases, at T = 0.1, 0.2, 0.4, 0.7, 1 after the initial and the steady state (——), respectively.

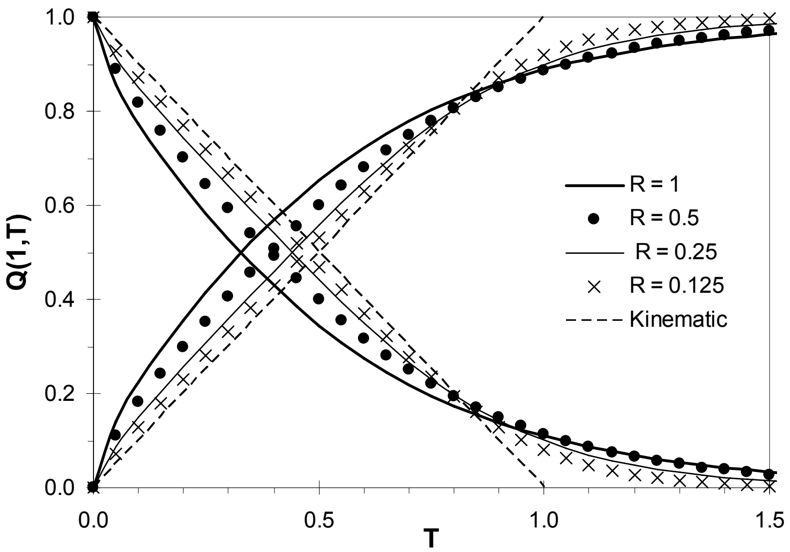

Fig. 4 Non-dimensional outflow hydrographs, $Q(1,T)$, in the build-up and drainage phases.

the solution tends to the KW approximation (and $h(0) \to 0$), while the flow turns more diffusive for $R > 0.2$ ($\eta_o \neq 0$). The best-fit relationship $\partial h/\partial x|_o/\tan\varphi \approx (1.09R - 0.33R^2)$ $\cos\varphi$ holds in the range $0 \leq R \leq 1$.

Verhoest & Troch (2000) used 999 to 9999 terms (more on steeper slopes) to evaluate their series solution. In contrast to a rigidly sequential evaluation, the series in equations (29) and (32) converge faster after collocation. Again, the lower the R value (kinematic tendency), the stiffer the profile and the more terms are needed (in Table 1, the exponents $-\lambda_i$ rise slowly for low R), especially at early times. For example, results for $R = 0.25$ at $T = 0.1$, computed with 50 and 10 terms, are identical to five decimal places, while the 50- and 20-term solutions agree to six decimal places ($k = 10, 20, 50$ collocation points spread evenly over the slope length, plus an endpoint, where the initial condition $Q|_{X=0} = 0$ or $H|_{X=1} = 0$ is satisfied). In contrast, for $R = 0.125$, the 50- and 10-term solutions agree only to one decimal place at $T = 0.1$; 20 collocation points give agreement to five decimal places. The larger number of series terms required as R decreases hints that the second-order term in equation (7), which controls the nature of the solution and its stiffness, progressively loses significance and that the KW model may be adequate.

ASSESSMENT OF THE QUASI-STEADY SOLUTION OF VERHOEST & TROCH

Next, the quasi-steady (henceforth QS) state method of Verhoest & Troch (2000) is examined for rapid calculations. That method references the transient outflow from r, $q(L,t')$, to a QS state by selecting a recharge rate r^* such that the transient depth profile at t' is fitted by the steady-state solution for $q(L,t') = r^*L$. The QS solution approximates the transient one well if the computational transition time between states, Δt, suffices for the flow to adjust. However, such an adjustment depends on the system parameters, as shown below for transitions in daily time steps (a typical increment in watershed modelling). For example, the simulation of the transient response by means of steady-state solutions requires appropriate temporal filtering of either data and/or model parameters (Adams & Koussis, 1980).

The parameters of the application example of Verhoest & Troch (2000) are: hydraulic conductivity: $K = 10^{-3}$ m s^{-1} = 86.4 m day^{-1}; drainable porosity: $n = 0.34$; angle: $\varphi = 2°$ or slope $S = 0.0349$; and length: $L = 100$ m. With these parameters, $\Delta t = 1$ day gives $\Delta T_1 = \Delta t \cdot KS/(nL) = 0.089$. Given the very high K value, the non-dimensional time for flow adjustment is short; thus $\Delta t = 1$ day in watershed modelling would be compatible with the QS approximation. But if the less extreme, yet still sizeable $K = 10^{-4}$ m s^{-1} = 8.64 m day^{-1} were used, all other parameters remaining unchanged, $\Delta T_1 = 0.0089$; this interval is likely insufficient for flow adjustment and the QS approach may be inappropriate in watershed modelling.

This is illustrated in an example with common parameter values: $K \approx 5.8 \times 10^{-5}$ m s^{-1} (5 m day^{-1}), $n = 0.25$, $\varphi = 10°$ ($S = 0.174$), $L = 100$ m, $r = 3.25$ mm h^{-1} and $r/K = 0.015$; thus $R = 0.53$ and $\eta_o = 0.125$ ($h_o \approx 2.17$ m). Initial condition is steady flow for $r = 0.25$ mm h^{-1} ($r/K = 0.001$; $R = 0.04$). In Fig. 5, the QS approximation of the transient profiles appears worsening as Δt decreases; $\Delta t = 1$ day corresponds to $\Delta T_1 = 0.034$. The QS solution inevitably fails at early times and approaches the new steady state gradually. This behaviour holds for all R; differences concern only the time needed for adjustment.

Verhoest & Troch (2000) derive an analytical solution—their equation (28)—on the premise of an initial steady flow for recharge rate r^*. That solution gives the

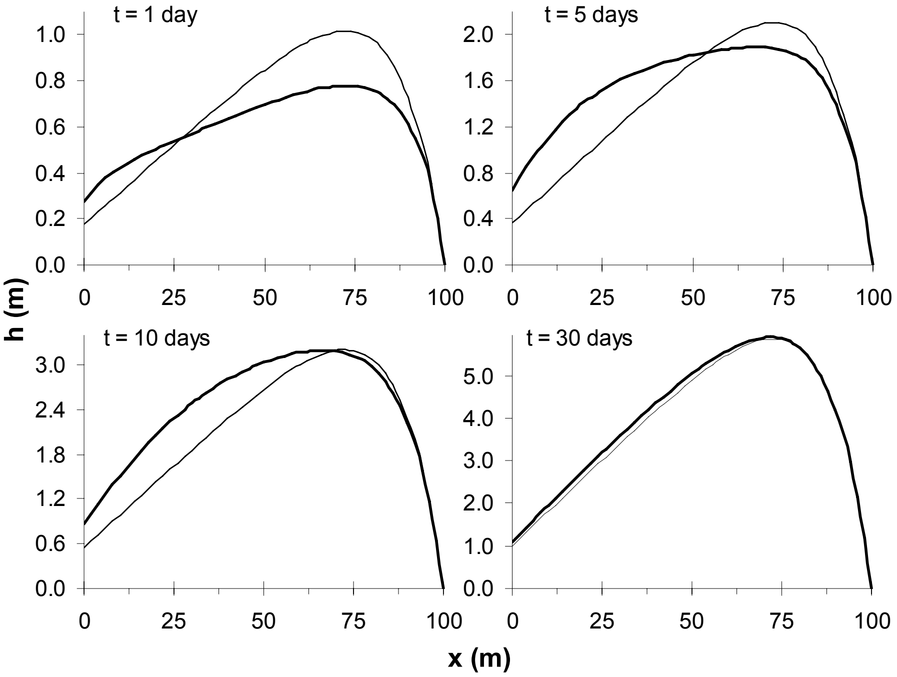

Fig. 5 Exact transient (——) and quasi-steady (——) depth solutions h (x,t).

response of the soil layer to a constant rate r, assuming the same linearization depth h_o for r^* and r. This assumption is acceptable if the recharge varies mildly, as in the example of Figure 6 in Verhoest & Troch (2000). By physical reasoning, however, $h_o = h_o(r)$, as h_o reflects the range of depths in a soil layer, which varies with the recharge; for example, $h_o \approx 0.5$ m if $0 < h < 1$ m, while $h_o \approx 3$ m if 1 m $< h < 5$ m, these ranges being associated with different r values.

Our new solution equation (29) described herein does not suffer from this restriction, because any initial condition can be enforced through the collocation procedure, even a transient flow from an arbitrary recharge history. [The two solutions agree ($\rho = 0$) when $r^* = 0$ and $h(x, t = 0) = 0$ (dry soil layer initially), provided h_o is the same.] A constant h_o makes the steady-state profiles similar, with the recharge as scaling factor; however, this behaviour conflicts with the closer approximation of a recharge-dependent h_o. For example, Fig. 6 shows that the non-dimensional locus of the profile maximum varies with the recharge R and that the profiles are not simply R-scalable.

Pauwels *et al.* (2002) solve equation (5) of this paper, with $r/K = 0$, for a steady flow initial condition, with a constant non-zero outflow depth; they elaborate that solution further to model baseflow. The basic solution equation (8) of Pauwels *et al.* (2002) differs from equation (28) of Verhoest & Troch (2000) only in the first exponential term, which reflects the constant outflow depth. It is again the constancy of h_o that allows the elegance of equations (25)–(30) of Pauwels *et al.* The use of h_o = const., regardless of recharge rate, is understandable from the computational perspective, for, if the linearization level were allowed to vary with r, the double-series in equation (28) of Pauwels *et al.* would have to be recalculated at each time step, greatly burdening the computation.

Basha & Maalouf (2005) and Koussis *et al.* (1998) have shown that the LAD model approximates experimental data well; it is thus argued that it is useful for

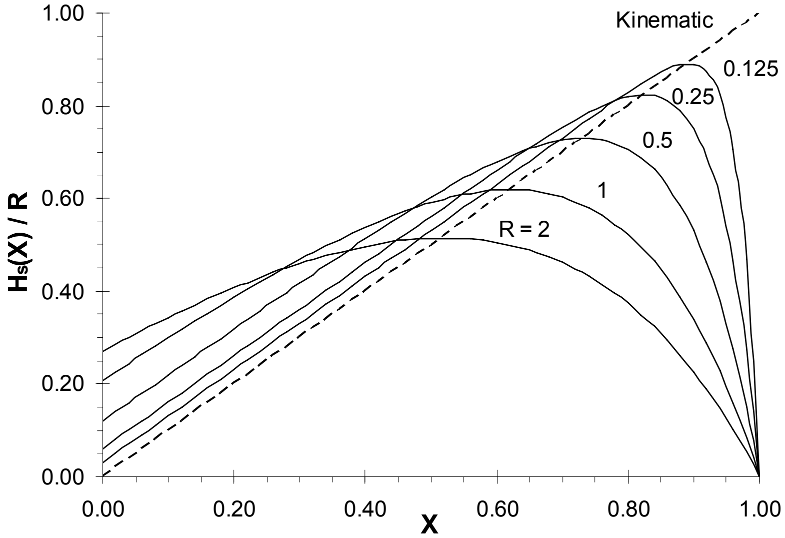

Fig. 6 Non-dimensional depth profiles at steady state.

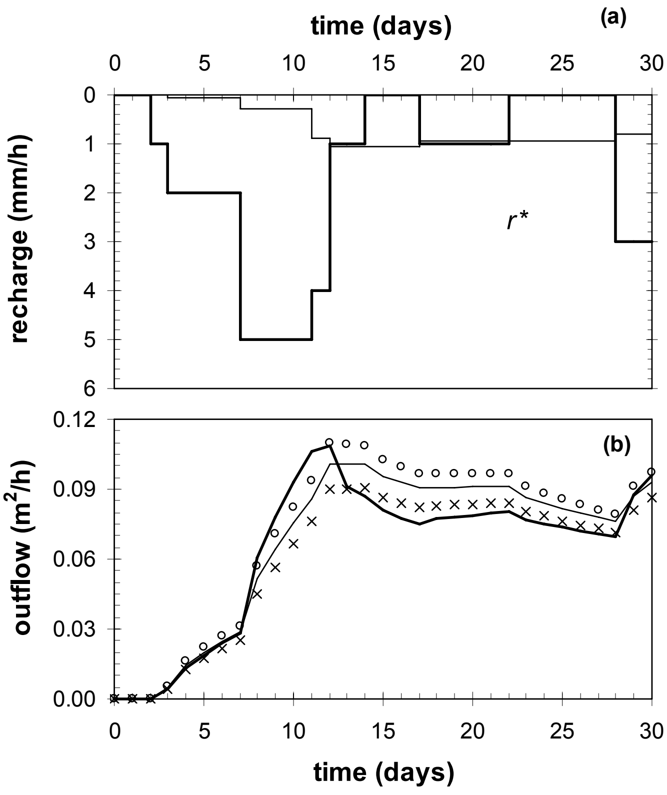

Fig. 7 Transient and quasi-steady-state outflow hydrographs $q(L,t)$ for a monthly recharge series: (a) recharge $r(t)$ (——) and reference rate $r^*(t)$ (——) for the optimal quasi-steady outflow; (b) exact transient flow for $h_o(r)$ (——) and quasi-steady-state flow, with h_o from r_{mean} = 1.533 mm h^{-1} (——), r = 1mm h^{-1} (×××) and r = 2mm h^{-1} (○ ○ ○).

applications, which Pauwels *et al*. (2002) show in the calculation of baseflow with their LAD-solution. The algorithm builds on the principle of superposition applied to a recharge pulse of duration τ: in the build-up phase $T \leq \tau$, the outflow is calculated by equation (33), and in the draining phase $T \geq \tau$, by the difference $Q(1,T) - Q(1,T-\tau)$. For a series of recharge pulses, the flow is given by the sum of the responses to the discrete pulses (recharge-drainage cycles). The transient solution and the QS solution of Verhoest & Troch (2000) are compared below for a monthly recharge series.

The parameters of the soil layer are $K = 2.1 \cdot 10^{-5}$ m s^{-1} (~1.8 m day^{-1}), $n = 0.25$, $\varphi = 15°$ ($S \approx 0.259$) and $L = 100$ m. The recharge ranges from 1 to 5 mm h^{-1}, or $0.2 \leq R \leq 1.1$, so the depths of linearization vary widely, 1.26 m $\leq h_o \leq$ 5.90 m [$0.5\eta_o(R)$ of equation (36)]. The QS method uses the same h_o throughout. The daily outflows are computed in the transient solution equation (33) with 25 terms and in the QS solution (equation (29) of Verhoest & Troch, 2000) with up to 250 terms (depending on temporal detail). Outflow starts at zero, following an extended dry period. The recharge, the outflow hydrographs and the r^* values of the QS solution are shown in Fig. 7. It is evident that the smoother QS solution cannot track recharge changes as well as the transient solution (which approximates nonlinearity via $h_o(r)$). In the present example, $q(L, t)$ and r^* vary markedly, in stark contrast to their near-constancy in the application example of Verhoest & Troch (2000) ($r^* \approx 3$ mm h^{-1} and 0.291 m^2 h^{-1} $\leq q(L,t) \leq$ 0.304 m^2 h^{-1}).

The QS solution with $h_o(r_{\text{mean}}$ = 46 mm/30 days = 1.533 mm h^{-1}) = 1.94 m ($R \approx 0.3$) is closest to the transient solution: mean-normalized RMS error, $RMSE$ = 15% and max. error = –20%. Figure 7 shows also the QS solutions for $h_o(r = 1$ mm h^{-1}) ($RMSE$ = 15%, max. error = –28%) and $h_o(r = 2$ mm h^{-1}) ($RMSE$ = 19%, max. error = 31%). Not shown is the very inferior QS solution ($RMSE$ = 40%) with $h_o(r)$ as in the transient solution. The transient and best QS profiles at t = 30 days are shown in Fig. 8. Importantly, the QS solution fails to conserve mass: over the month, the difference between total recharge and total outflow does not equal the change of water volume in the soil. In this example, the discrepancy is only ~2% for the best QS solution, but the error rises for a non-optimal linearization, e.g. to –6.5% and 7% for $h_o(r = 1$ mm h^{-1}) and $h_o(r = 2$ mm h^{-1}), respectively. The sensitivity of results to the h_o value is of concern, because the optimal h_o is unknown *a priori* (it depends also on future recharges).

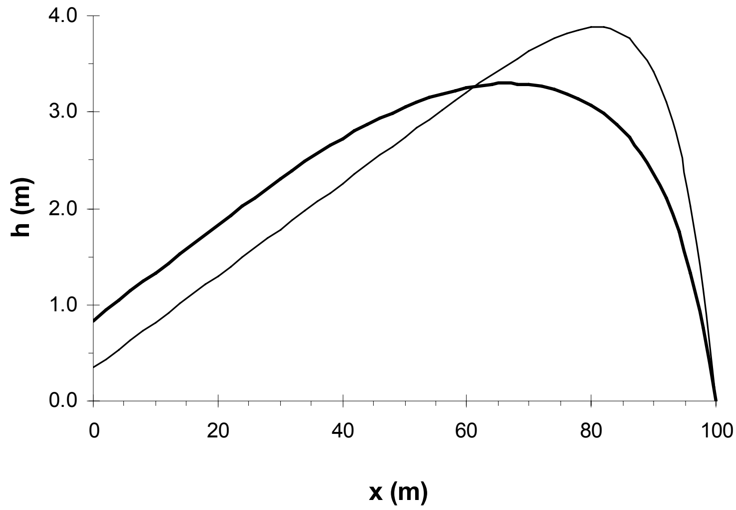

Fig. 8 Transient and optimal quasi-steady depth profiles, at t = 30 days.

SUMMARY AND CONCLUSIONS

Planar subsurface hillslope flow has been studied on the basis of the linearized extended 1-D Boussinesq equation (Dupuit-Forchheimer approximation; LAD model). The present LAD model includes the recharge components normal and parallel to the inclined bed; the latter is often ignored, yet this simplification is justified only if $r/K << 1$. A closed-form solution has been postulated as the sum of the steady-state and the transient (series) solutions, the latter in a separation-of-variables form. The parameters of the transient series depend on the dimensionless linearization depth $\eta_o(R) = h_o(R)\cos\varphi/L\sigma$, $R = r/K\sigma^2$ and are calculated by iteration. The coefficients of the series are determined by collocating the general solution function with an arbitrary initial condition at a limited number of points, which also limits the number of series terms and makes the solution efficient.

The non-dimensional linearization parameter $\eta_o(R)$ emerges as central in the soil layer's response. The series converge rapidly for large η_o (diffusive flow) and slowly as $\eta_o \rightarrow 0$ (KW). Also, $\eta_o(R)$, not the slope, properly characterizes the flow as kinematic or diffusive; $R = 0.2$ is an approximate kinematic-diffusive limit. As a consequence of the $\eta_o(R)$ dependence, the flow in a soil layer can be more or less kinematic or diffusive according to the recharge, while η_o cannot be a constant aquifer schematization parameter.

The new analytical solution can be used to study subsurface stormflow/baseflow in watershed modelling, as well as leachate flow towards lateral collection drains in landfills and may also serve as benchmarking standard of approximate solutions. Here, it has been used to test the QS state method of Verhoest & Troch (2000). First, it was reasoned that the QS approximation holds, if the computing time step exceeds the system-dependent transition time between flow states. The validity of this argument has been illustrated in an example, with common soil parameter values, in which the QS depth profile approaches steady state for $\Delta t \approx 20$ days >> 1 day, failing to approximate earlier transient states adequately.

Transient and QS daily outflows were simulated in an example with a 30-day recharge series. The comparison made evident that the QS solution ($RMSE$ = 15%, max. error = –20%) cannot track recharge changes as well. Importantly, the QS solution fails to conserve mass. The error is ~2% for the optimal h_o, but rises rapidly for other h_o values; the sensitivity to the h_o value is of concern because the optimal h_o is unknowable beforehand.

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